Math 102 Homework-2 Solutions
Due Date: 23 July 2008 Wednesday
Either hand in your homework solutions in class or put them in my mail box until 17:00 on Wednesday.

Q-1) Show that the vector field

$$\mathbf{F} = \big(-\tan(x+y^2+z^3),\ -2y\tan(x+y^2+z^3),\ -3z^2\tan(x+y^2+z^3)\big)$$

is conservative. Find a potential function for $\mathbf{F}$ and evaluate the integral

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F}\cdot\mathbf{T}\, d\sigma.$$

Solution: A potential function for this field is $f = \ln\cos(x+y^2+z^3)+C$. The integral then has the values $f(1,2,3)-f(0,0,0) = \ln\cos 32 \approx -0.181$.

Q-2) let $\mathbf{F} = \left(\dfrac{2x}{x^2+y^4}, \dfrac{4y^3}{x^2+y^4}\right)$. Evaluate $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\, ds$, where C is the circle of radius R centered at the origin. Beware here that the Green's theorem does not hold since $\mathbf{F}$ is not defined at the origin. Observe that in this problem $M_y = N_x$ for the vector field $\mathbf{F} = (\mathbf{M}, \mathbf{N})$. Suppose you have the task of providing such vector fields on demand. How would you construct such vector fields without much effort? How did I *invent* the above vector field?

Solution: Let R_ϵ be the rectangle formed by the lines $x = \pm\epsilon$ and $y = \pm\epsilon$ where $\epsilon > 0$ is small enough that the rectangle R_ϵ totally lies inside C. Then Green's theorem applies to the region bounded by C and R_ϵ and since $M_y = N_x$, where $\mathbf{F} = (M, N)$, we must have

$$\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_{R_\epsilon} \mathbf{F}\cdot\mathbf{T}\, ds$$

where R_ϵ is positively oriented. But using the obvious parametrization for each side of R_ϵ, we find that $\int_{R_\epsilon} \mathbf{F}\cdot\mathbf{T}\, ds = 0$ since on each side we are integrating an odd function on $[-1, 1]$. This gives

$$\int_C \mathbf{F}\cdot\mathbf{T}\, ds = 0.$$

To generate such vector fields, start with any function and calculate its gradient. For example the above vector field is the gradient of $\ln(x^2+y^4)$. If the function is not defined at the origin, then its gradient also fails to be defined there.

Q-3) Find the area of the surface S cut from the cone $z^2 = 4x^2 + 4y^2$, $z \geq 0$, by the cylinder $x^2 + y^2 = 2x$.

Solution: Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 2x\ \}$, and let $f = 4x^2 + 4y^2 - z^2$. Then the surface is given by $f = 0$ over D.

$d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dA = \sqrt{5} dA$. Thus

$$\text{Surface area} = \int_S d\sigma = \int_D \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dA = \sqrt{5} \int_D dA = \sqrt{5}\,\pi.$$

Q-4) Evaluate the integral

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\; d\sigma$$

where S is the level surface given by $x^2 + z^2 - 4(x + z) - y + 8 = 0$, $0 \leq y \leq 4$, and

$$\mathbf{F} = \left(x^2 z + \ln(y^2 + 1),\ \cosh(x^2 + y^2) - \ln(z^2 + 1),\ \frac{y^3}{y^2 + 1} - xz^2 \right).$$

Solution: Let $D = \{(x, z) \in \mathbb{R}^2 \mid (x - 2)^2 + (z - 2)^2 \leq 4\ \}$ and let the boundary of D be C. Then applying Stokes' theorem twice we get

$$\begin{aligned} \int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\; d\sigma &= \int_C \mathbf{F} \cdot \mathbf{T}\; ds \\ &= \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1}\; d\sigma \end{aligned}$$

where $\mathbf{n_1}$ is the unit normal of D pointing towards y-direction to be compatible with the orientation on C which in turn is induced by $\mathbf{n}$. Thus $\mathbf{n_1} = \mathbf{j}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n_1} = x^2 + z^2$. This gives

$$\begin{aligned} \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1}\; d\sigma &= \int\int_D (x^2 + z^2)\; dxdz \\ &= \int_0^4 \int_{2-\sqrt{4z-z^2}}^{2+\sqrt{4z-z^2}} (x^2 + z^2)\; dxdz \\ &= 40\pi, \end{aligned}$$

where the last line should be obtained through a computer algebra system.

An easy way to evaluate this integral by hand is to make the change of variables $X = x - 2$ and $Z = z - 2$, and change to polar coordinates in the new XZ system. This gives the easy integral

$$\int_0^{2\pi} \int_0^2 (r^2 + 4r(\cos\theta + \sin\theta) + 8) r dr d\theta = 40\pi.$$

Another way to solve this problem is to use Stokes' theorem only once. Then we parameterize the boundary of S as

$$\mathbf{r}(\theta) = (2 + 2\sin\theta,\ 2,\ 2 + 2\cos\theta), \quad \theta \in [0, 2\pi].$$

Notice how the correct orientation of the boundary is provided by the parametrization. Now we have

$$d\mathbf{r} = (2\cos\theta,\ 0,\ -2\sin\theta)\ d\theta,$$

and $\mathbf{F}\cdot d\mathbf{r}$ then becomes

$$(2\cos\theta)\left((2+2\sin\theta)^2(2+2\cos\theta)+\ln 17\right)+(-2\sin\theta)\left(\frac{64}{17}+(2+2\sin\theta)(2+2\cos\theta)^2\right).$$

Integrating this we get

$$\int_0^{2\pi}\mathbf{F}\cdot d\mathbf{r} = 40\pi.$$

Finally, you may try to evaluate the integral as is. Then you will get

$$\nabla\times\mathbf{F} = [3\,\frac{y^2}{y^2+1} - 2\,\frac{y^4}{\left(y^2+1\right)^2} + 2\,\frac{z}{z^2+1}, x^2+z^2, 2\,\sinh\left(x^2+y^2\right)x - 2\,\frac{y}{y^2+1}].$$

If $f = x^2+z^2-4(x+z)-y+8$, then the unit outward normal of S is in the direction of $-\nabla f$ where

$$\nabla f = [2\,x-4, -1, 2\,z-4].$$

You will have

$$\nabla\times\mathbf{F}\cdot\mathbf{n}\ d\sigma = \nabla\times\mathbf{F}\cdot\nabla f dxdz$$

and the integral will be evaluated over the disk $(x-2)^2+(z-2)^2\leq 4$. You will also need to substitute y with $x^2+z^2-4(x+z)+8$ since the integrand lives on the surface S.

Putting these together, you will end up with the double integral

$$\int_0^4\int_{2-\sqrt{4z-z^2}}^{2+\sqrt{4z-z^2}}\left\{\left[3\,\frac{\left(x^2+z^2-4\,x-4\,z+8\right)^2}{\left(x^2+z^2-4\,x-4\,z+8\right)^2+1} - 2\,\frac{\left(x^2+z^2-4\,x-4\,z+8\right)^4}{\left(\left(x^2+z^2-4\,x-4\,z+8\right)^2+1\right)^2}\right.\right.$$
$$\left. +2\,\frac{z}{z^2+1}\right]\left(-2\,x+4\right)+x^2+z^2$$
$$+\left[2\,\sinh\left(x^2+\left(x^2+z^2-4\,x-4\,z+8\right)^2\right)x\right.$$
$$\left.\left. -2\,\frac{x^2+z^2-4\,x-4\,z+8}{\left(x^2+z^2-4\,x-4\,z+8\right)^2+1}\right]\left(-2\,z+4\right)\right\}\ dxdz.$$

An attempt to numerically evaluate this on Maple will give, after a long pause, 125 which is almost $40\pi\approx 125.6$.

Q-5) Solve the very last problem of the book, exercise 21 on page 1228:
Show that the volume of a region D in space enclosed by the oriented surface S with outward normal $\mathbf{n}$ satisfies the identity

$$V = \frac{1}{3} \int\int_S \mathbf{r} \cdot \mathbf{n} \, d\sigma,$$

where $\mathbf{r}$ is the position vector of the point (x, y, z) in D.

Solution: Taking $\mathbf{r}$ as the vector field $\mathbf{F}$ of the divergence theorem, we find that

$$\int\int_S \mathbf{r} \cdot \mathbf{n} \, d\sigma = \int\int\int_D \nabla \cdot \mathbf{r} \, dV = \int\int\int_D 3 \, dV = 3V,$$

verifying the required equality.

Please send comments and questions to sertoz@bilkent.edu.tr