Due Date: July 19, 2010, Monday NAME:..
Time: 10:30
Ali Sinan Sertöz STUDENT NO:..

Math 102 Calculus II – Homework II – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1) For any $h \geq 0$ consider the region R_h in $\mathbb{R}^3$ bounded by the surfaces $z = (y+1)x^2$, $y = 0$, $y = 1$ and $z = h$. Find the volume of R_h.

Solution:

$$\begin{aligned} Volume &= 2\int_0^1 \int_0^{\sqrt{h/(y+1)}} \int_{(y+1)x^2}^{h} dz\,dx\,dy \\ &= 2\int_0^1 \int_0^{\sqrt{h/(y+1)}} h-(y+1)x^2\,dx\,dy \\ &= \frac{4h^{3/2}}{3}\int_0^1 \frac{dy}{\sqrt{y+1}} \\ &= \frac{8h^{3/2}}{3}(\sqrt{2}-1) \\ &\approx (1.104)h\sqrt{h}. \end{aligned}$$

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Q-2) Let R be the region in $\mathbb{R}^3$ in the first octant bounded by the coordinate planes and the unit sphere. Evaluate the integral of the function $e^{(x^2+y^2+z^2)^{3/2}}$ on R.

Solution:

The problem requires that we pass to spherical coordinates.

$$\begin{aligned}
\iiint_R e^{(x^2+y^2+z^2)^{3/2}} dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{\rho^3}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{3}\sin\phi \, e^{\rho^3}\Big|_{\rho=0}^{\rho=1}\right) d\phi \, d\theta \\
&= \frac{e-1}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta \\
&= \frac{e-1}{3}\int_0^{\pi/2}\left(-\cos\phi\Big|_0^{\pi/2}\right) d\theta \\
&= \frac{e-1}{3}\int_0^{\pi/2} d\theta \\
&= \frac{(e-1)\pi}{6} \\
&\approx 0.899
\end{aligned}$$

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Q-3) Consider the vector field $\vec{F} = \left(\frac{1}{x + y^2 + z^3}, \frac{2y}{x + y^2 + z^3} + 1, \frac{3z^2}{x + y^2 + z^3} + 2z \right)$.
Calculate the work done by $\vec{F}$ along the path $C = C_1 + C_2 + C_3$.

C_1 is along the semicircle in the yz-plane with center at the origin and radius 2. C_1 follows this semicircle from $(0, -2, 0)$ towards $(0, 2, 0)$ with $z \geq 0$.
C_2 goes from $(0, 2, 0)$ towards the point $(2, 1, 0)$ along the ellipse $\frac{3x^2}{16} + \frac{y^2}{4} = 1$ in the xy-plane.
C_3 goes from the point $(2, 1, 0)$ towards the point $(2, 1, 1)$ along a straight line.

Solution:

For the problem to be *reasonable*, $\vec{F}$ must be conservative! In fact we find that

$$\vec{F} = \nabla f, \quad \text{where} \quad f = \ln(x + y^2 + z^3) + y + z^2,$$

and

$$\text{Work along } C = \int_C \vec{F} \cdot dr = f(2, 1, 1) - f(0, -2, 0) = (\ln 4 + 2) - (\ln 4 - 2) = 4.$$

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Q-4) Consider the curve of intersection of the surfaces $z = y$ and $z = x^2 + y^2$, and let C be the path on this curve from the origin to the point $(0, 1, 1)$ lying in the first octant. Calculate the work done by the vector $\vec{F} = (x, x^2, y + z)$ on the path C.

Solution:

A parametrization of the path C is $\vec{r}(t) = (\sqrt{t(1-t)}, t, t)$, $0 \le t \le 1$.

$$\begin{aligned}
\text{Work along } C &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_0^1 \vec{F}(\vec{r}(t)) \cdot d\vec{r}(t) \\
&= \int_0^1 (\sqrt{t - t^2}, t - t^2, 2t) \cdot (\frac{1-2t}{2\sqrt{t-t^2}}, 1, 1)\, dt \\
&= \int_0^1 (\frac{1}{2} + 2t - t^2)\, dt \\
&= \frac{7}{6}.
\end{aligned}$$