Date: June 19, 2010, Saturday NAME:..

Time: 10:30-12:30

Ali Sinan Sertöz STUDENT NO:..

Math 102 Calculus II – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Use at your own risk!

$$
\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
\tan(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\sec(x) &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\tan^{-1}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\end{aligned}
$$

NAME: STUDENT NO:

Q-1) Show that $\lim_{n\to\infty} a_n = 0$, where $a_n = \dfrac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}{4\cdot 9\cdot 14\cdots(5n-1)}$.

Solution:

We have two solutions to this problem. First the straightforward solution.

Observe that $\dfrac{2n+1}{5n-1} \le \dfrac{1}{2}$ for all $n \ge 3$. This gives

$$0 < a_n = \left(\frac{3\cdot 5}{4\cdot 9}\right)\left(\frac{7\cdots(2n+1)}{14\cdots(5n-1)}\right) \le \left(\frac{5}{12}\right)\left(\frac{1}{2}\right)^{n-2}.$$

By the Sandwich theorem, we have $\lim_{n\to\infty} a_n = 0$.

The other solution requires a little imagination. First observe that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2n+3}{5n+4} = \frac{2}{5} < 1,$$

so $\sum_{n=1}^{\infty} a_n$ converges, forcing $\lim_{n\to\infty} a_n = 0$.

NAME: STUDENT NO:

Q-2-a) Check the following series for converge:

$$\sum_{n=1}^{\infty} \frac{\ln n}{(19n^2 + 6n + 2008)}$$

Solution:

Limit compare with $\sum \frac{\ln n}{n^2}$ which converges by the integral test, to conclude that the given series converges.

Use integration by parts to integrate $\frac{\ln x}{x^2}$ as follows: Set $u = \ln x$ and then

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = -\left(\frac{\ln x}{x} \Big|_1^{\infty} \right) + \int_1^{\infty} \frac{dx}{x^2} = -\left(\frac{1}{x} \Big|_1^{\infty} \right) = 1.$$

Q-2-b) Check the following two series for convergence: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ and $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution:

For the first series

$$\frac{a_{n+1}}{a_n} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1 \quad \text{as} \quad n \to \infty.$$

Therefore the first series converges by the ratio test. Hence the general term goes to zero as n goes to infinity. The general term of the second series is the reciprocal of the general term of the first series and hence goes to infinity as n goes to infinity. Then the second series diverges by the divergence test.

NAME: STUDENT NO:

Q-3) Find the sum

$$\sum_{n=2}^{\infty} \frac{6}{(n-1)(n)(n+2)}$$

Solution:

$$\frac{6}{(n-1)(n)(n+2)} = \frac{-3}{n} + \frac{2}{n-1} + \frac{1}{n+2}.$$

Adding these from $n = 2$ to $n = k$ we find

$$s_k = \sum_{n=2}^{k} \frac{6}{(n-1)(n)(n+2)} = \frac{7}{6} - \frac{4+3k}{k(k+1)(k+2)}.$$

Hence the sum is $\lim_{k\to\infty} s_k = \frac{7}{6}$.

NAME: STUDENT NO:

Q-4) Let r be the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{7n^2+1}{3n^3+n+81} x^n$.

a) Find r. *(10 points)*

b) Check the convergence of the series for $x = r$ and for $x = -r$. *(5+5 points)*

Solution:

Let $a_n(x) = \frac{7n^2+1}{3n^3+n+81} x^n$ and use ratio test for the absolute values. $\frac{|a_{n+1}(x)|}{|a_n(x)|} \to |x|$ as $n \to \infty$.

For absolute convergence we must have $|x| < 1$. So the radius of convergence is 1.

When $x = 1$, we have $\frac{|a_n(x)|}{1/n} \to 7/3$ as $n \to \infty$. Hence the series diverges at $x = 1$ by limit comparing with the Harmonic series.

When $x = -1$, we have an alternating series. The general term goes to zero and its absolute value decreases (see its first derivative below). Then the series converges at $x = -1$ by the alternating series test.

To see that the general term decreases to zero, let $f(t) = \frac{7t^2+1}{3t^3+t+81}$. Then $f'(t) = \frac{-21t^4 - 2t^2 + 1134t - 1}{(3t^3+t+81)^2}$ which is negative for all large t (in fact for all $t \geq 4$.)

NAME: STUDENT NO:

Q-5) Find

$$\lim_{x\to 0} \frac{6(\tan x)(\sec x) - 6x - 5x^3}{(e^x - 1 - x)(\sin x - x)}.$$

Solution:

Using the Taylor expansions of the functions involved in the limit, we have

$$\frac{6(\tan x)(\sec x) - 6x - 5x^3}{(e^x - 1 - x)(\sin x - x)} = \frac{\frac{61}{20}\,x^5 + \frac{277}{168}\,x^7 + \cdots}{-\frac{1}{12}\,x^5 - \frac{1}{36}\,x^6 + \cdots} \to -\frac{183}{5} \quad \text{as} \quad n \to \infty.$$