Date: July 10, 2010, Saturday NAME:..
Time: 10:30-12:30
Ali Sinan Sertöz STUDENT NO:..

Math 102 Calculus II – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

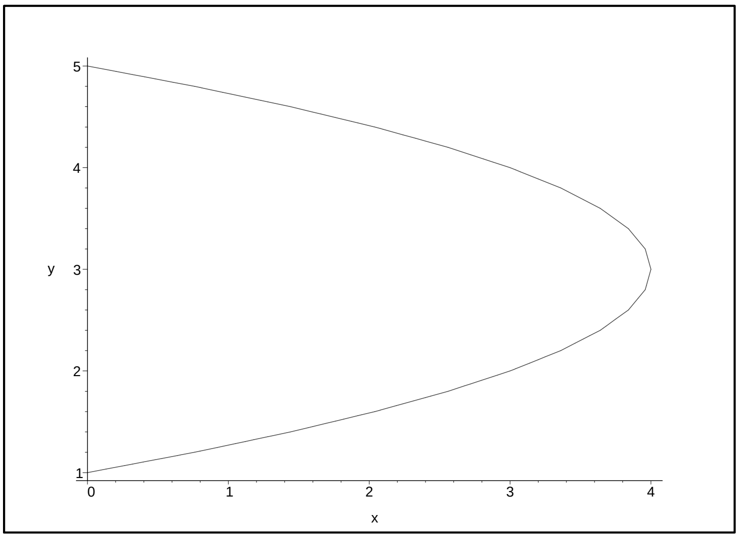
Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail, if that is necessary. A correct answer without proper reasoning may not get any credit, except in the fill-in-the-blanks type problems.

Q-1) Let R be the region in the plane bounded by the curve $(y-3)^2+x=4$ and the y-axis. Sketch this region and write the limits of integration in the following integrals, into the given boxes.
(Grading: sketch=4 points, each correctly filled box=2 points.)

$$\int_{\square}^{\square}\int_{\square}^{\square} dx\,dy = \int_{\square}^{\square}\int_{\square}^{\square} dy\,dx.$$

Solution:



$$\int_{1}^{5}\int_{0}^{4-(y-3)^2} dx\,dy = \int_{0}^{4}\int_{3-\sqrt{4-x}}^{3+\sqrt{4-x}} dy\,dx.$$

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Q-2 Consider the function $f(x, y) = x^3 + x^2 + y^2 - 4xy$ defined on $\mathbb{R}^2$. Find its critical points and classify the critical points using the second derivative test. Decide if the function has global maximum and minimum values.
(Grading: critical points=9 points, Second derivative test=9 points, global min/max=2 points.)

Solution: $f_x = 3x^2 + 2x - 4y = 0$, $f_y = 2y - 4x = 0$.
$f_y = 0$ gives $y = 2x$. Putting this into $f_x = 0$ gives $x = 0$ or $x = 2$.
Hence the critical points are $(0, 0)$ and $(2, 4)$.

$f_{xx} = 6x + 2$, $f_{yy} = 2$, $f_{xy} = -4$. $\nabla = f_{xx}f_{yy} - f_{xy}^2 = 12x - 12$.
$\nabla(0, 0) = 12 < 0$, so $(0, 0)$ is a saddle point.
$\nabla(2, 4) = 12 > 0$ and $f_{yy}(2, 4) = 2 > 0$, so $(2, 4)$ is a local minimum point.

Since the dominating term in f is x^3, the function is unbounded.

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Q-3) Let R be the region in $\mathbb{R}^3$ lying in the first octant, i.e. $x, y, z \geq 0$, and bounded by the cylindrical surfaces $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. Write the limits of integration into the following boxes and evaluate the integral.
(Grading: each box=1 point, evaluation=8 points.)

$$\int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} dz\,dy\,dx = \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} dy\,dx\,dz.$$

Solution:

$$\int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{\sqrt{1-x^2}}} \int_{\boxed{0}}^{\boxed{\sqrt{1-x^2}}} dz\,dy\,dx$$

$$= \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{\sqrt{1-z^2}}} \int_{\boxed{0}}^{\boxed{\sqrt{1-x^2}}} dy\,dx\,dz.$$

The first integral is easier to evaluate:

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx \\
&= \int_0^1 \left(y\sqrt{1-x^2} \Big|_0^{\sqrt{1-x^2}} \right) dx \\
&= \int_0^1 (1-x^2)\,dx \\
&= \left(x - \frac{x^3}{3} \Big|_0^1 \right) \\
&= \frac{2}{3}.
\end{aligned}$$

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Q-4) Assume that the equation $x^2+y^3+xy+x^2z+yz^2+z^5=11$ defines z as a C^∞ function of x and y around the point $(x,y,z)=(1,2,-1)$.
a) Write the equation of the tangent plane, at the point $(x,y,z)=(1,2,-1)$, to the surface defined by the above equation. *(10 points)*
b) Calculate the value of z_{xx} at the point $(x,y,z)=(1,2,-1)$. *(10 points)*

Solution:

Let $f(x,y,z)=x^2+y^3+xy+x^2z+yz^2+z^5-11$. $\nabla f(1,2,-1)=(2,14,2)$. The equation of the tangent plane at the given point is

$$(2,14,2)\cdot(x-1,x-2,x+1)=0,$$

which gives

$$x+7y+z=14.$$

For the second part, differentiating f woth respect to x and treating z as a function of x and y gives

$$(2x+y+2xz)+(x^2+2yz+5z^4)\,z_x=0. \qquad (*)$$

Substituting $(x,y,z)=(1,2,-1)$ gives

$$z_x=-1.$$

Now differentiating both sides of $(*)$ again with respect to x we get

$$(2+2z+2xz_x)+(2x+2yz_x+20z^3z_x)\,z_x+(x^2+2yz+5z^4)\,z_{xx}=0$$

and substituting $(x,y,z)=(1,2,-1)$ and $z_x=-1$ we get

$$z_{xx}=10.$$

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Q-5) Find the maximum and minimum values of $f(x, y, z) = x + \frac{y}{2} + \frac{z}{3}$ on the sphere $x^2 + y^2 + z^2 = (42)^2$.

Solution:

$\nabla f = (1, \frac{1}{2}, \frac{1}{3}) = \lambda\,(x, y, z)$ gives $x = \frac{1}{\lambda}$, $y = \frac{1}{2\lambda}$ and $z = \frac{1}{3\lambda}$.

Putting these into the equation of the sphere we get $\lambda^2 = \frac{1}{(36)^2}$. Hence $\lambda = \pm\frac{1}{36}$, and $(x, y, z) = \pm(36, 18, 12)$.

The maximum value is $f(36, 18, 12) = 49$ and the minimum value is $f(-36, -18, -12) = -49$.