Quiz # 8
Math 102-Section **06** Calculus II
13 April 2017, Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Bilkent University

Your Name: ....

Student ID: .... Your Department: ....

Show your work in detail. Correct answers without justification are never graded.

Q-1) Define a sequence $(a_n)_{n=1}^{\infty}$ recursively as $a_1 > 6$, and $a_{n+1} = \dfrac{a_n}{2} + \dfrac{18}{a_n}$.

(i) Show that all $a_n > 6$.

(ii) Assuming (i), show that the sequence is decreasing.

(iii) Assuming (i)-(ii), show that the sequence is convergent.

(iv) Assuming (i)-(ii)-(iii), find the limit of the sequence.

Answer: (i) We will use induction on n. We already know that $a_1 > 6$. Assume that $a_n > 6$ for some $n \geq 1$. Then $a_{n+1} = \dfrac{a_n}{2} + \dfrac{18}{a_n} > 2\sqrt{\dfrac{a_n}{2}\dfrac{18}{a_n}} = 6$, where we used the arthmetic-geometric mean inequality $\frac{x+y}{2} \geq \sqrt{xy}$ where $x, y \geq 0$ and equality holding if and only if when $x = y$. Thus by induction we showed that $a_n > 6$ for all $n \geq 1$.

(ii) $a_n - a_{n+1} = a_n - \dfrac{a_n}{2} - \dfrac{18}{a_n} = \dfrac{a_n^2 - 36}{2a_n} > 0$ by (ii). Hence the sequence is decreasing.

(iii) The sequence is monotone and bounded, so is convergent.

(iv) Let the limit be L which exists by (iii). Use the equation $a_{n+1} = \dfrac{a_n}{2} + \dfrac{18}{a_n}$, and take the limit of both sides as n goes to infinity. Solve the resulting equation for L to find that $L = \pm 6$. Since all $a_n > 0$, we must then have $L = 6$.