1a. In this part, just fill in the boxes. No explanation is required. No partial points will be given.

❶ Give an example of a plane that contains the x-axis, but does not contain the y- and z-axes by writing its equation in the box below. [The box should contain nothing except the equation!]

$$\boxed{y+z=0}$$

❷ Give an example of a line that does not intersect the xy-plane, but intersects each of the yz- and xz-planes at exactly one point by writing its parametric equations in the box below.

[The box should contain nothing except the parametric equations!]

$$\boxed{\begin{aligned} x&=t\\ y&=t\\ z&=1\end{aligned}}$$

1b. The positions of two points P_1 and P_2 in the space as a function of time t are given by:

$$\mathbf{r}_1=\overrightarrow{OP_1}=(4t-1)\,\mathbf{i}+t^2\,\mathbf{j}+t\,\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_2=\overrightarrow{OP_2}=3t\,\mathbf{i}+t\,\mathbf{j}+t^3\,\mathbf{k}$$

Find all times t when there is a plane $\mathcal{P}$ such that

- The plane $\mathcal{P}$ passes through the points P_1 and P_2 at time t, and
- The velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points P_1 and P_2 at time t are parallel to the plane $\mathcal{P}$.

$\circledast \iff \overrightarrow{P_1P_2}, \vec{v}_1, \vec{v}_2 \parallel \mathcal{P} \iff \overrightarrow{P_1P_2} \perp \vec{v}_1 \times \vec{v}_2 \iff \overrightarrow{P_1P_2}\cdot(\vec{v}_1\times\vec{v}_2)=0$

$\vec{v}_1 = 4\vec{i}+2t\vec{j}+\vec{k}$, $\quad \vec{v}_2 = 3\vec{i}+\vec{j}+3t^2\vec{k}$

$$\vec{v}_1\times\vec{v}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 2t & 1\\ 3 & 1 & 3t^2\end{vmatrix} = (2t\cdot 3t^2-1\cdot 1)\vec{i}-(4\cdot 3t^2-3\cdot 1)\vec{j}+(4\cdot 1-3\cdot 2t)\vec{k}$$
$$=(6t^3-1)\vec{i}+(3-12t^2)\vec{j}+(4-6t)\vec{k}$$

$\cdot\ \overrightarrow{P_1P_2}=(1-t)\vec{i}+(t-t^2)\vec{j}+(t^3-t)\vec{k}$

$\circledast \iff (1-t)\cdot(6t^3-1)+(t-t^2)\cdot(3-12t^2)+(t^3-t)\cdot(4-6t)=0$

$\iff (t-1)\cdot(-6t^3+1-3t+12t^3+4t^2+4t-6t^3-6t^2)=0$

$\iff (t-1)\cdot(2t^2-t-1)=0 \iff (t-1)\cdot(t-1)\cdot(2t+1)=0 \iff t=1 \text{ or } t=-\frac{1}{2}$

The conditions $\circledast$ are satisfied exactly when $t=1$ or $t=-\frac{1}{2}$.

2. The *Bateman-Burgers equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,\frac{\partial u}{\partial x}$$

arises in the study of nonlinear acoustics and gas dynamics in fluid mechanics, and in the study of traffic flow in civil engineering.

Find all possible values of the pair of constants (a, b) for which the function

$$u(x,t) = \frac{x}{ax^2 + bt + 1}$$

satisfies the Bateman-Burgers equation for all (x,t) with $ax^2 + bt + 1 \neq 0$.

$$u_t = -bx \cdot (ax^2+bt+1)^{-2}$$

$$u_x = (ax^2+bt+1)^{-1} - x \cdot 2ax \cdot (ax^2+bt+1)^{-2}$$

$$u_{xx} = -2ax\,(ax^2+bt+1)^{-2} - 4ax\,(ax^2+bt+1)^{-2} + 2x \cdot (2ax)^2 \cdot (ax^2+bt+1)^{-3}$$

Hence:

$u_t = u_{xx} + u \cdot u_x$ for all (x,t) with $ax^2+bt+1 \neq 0$

$$\Updownarrow$$

$$-bx \cdot (ax^2+bt+1)^{-2} = -6ax \cdot (ax^2+bt+1)^{-2} + 8a^2x^3\,(ax^2+bt+1)^{-3}$$
$$+ x \cdot (ax^2+bt+1)^{-2} - 2ax^3 \cdot (ax^2+bt+1)^{-3}$$

for all (x,t) with $ax^2+bt+1 \neq 0$

$$\Updownarrow$$

$$(6a-b-1)(ax^2+bt+1) = 2a(4a-1)x^2 \quad \text{for all } (x,t)$$

$$\Updownarrow$$

$$\left.\begin{cases} 6a-b-1 = 0 \\ (6a-b-1)\cdot b = 0 \\ (6a-b-1)\cdot a = 2a\cdot(4a-1) \end{cases}\right\} \iff 6a-b-1=0 \text{ and } a(4a-1)=0$$

$$\iff \left(a=0 \text{ and } b=-1\right) \text{ or } \left(a=\tfrac{1}{4} \text{ and } b=\tfrac{1}{2}\right)$$

$$\iff (a,b) = (0,-1) \text{ or } \left(\tfrac{1}{4}, \tfrac{1}{2}\right)$$

3a. You are given the following information about a differentiable function $f(x, y)$:

① The tangent line to the level curve $f(x, y) = f(1, -1)$ at the point $(1, -1)$ has the equation $5x - 8y = 13$.

② $\left.\dfrac{d}{dt} f(t^3 - 2t^2 + 1, t^2 - 4/t - 3)\right|_{t=2} = 1$.

Choose one of the following:

☐ The given data is not consistent.

☐ The given data is consistent, but not sufficient to determine $\nabla f(1, -1)$.

☑ The given data is consistent, and sufficient to determine $\nabla f(1, -1)$.

Now prove your claim.

(1) $\Rightarrow$ $\vec{\nabla} f(1,-1) = c \cdot (5\vec{i} - 8\vec{j})$ for some scalar c.

(2) $\Rightarrow$ $1 = f_x(1,-1) \cdot \left.\frac{d}{dt}(t^3-2t^2+1)\right|_{t=2} + f_y(1,-1) \cdot \left.\frac{d}{dt}\left(t^2 - \frac{4}{t} - 3\right)\right|_{t=2}$

$= 5c \cdot (3t^2 - 4t)\big|_{t=2} + (-8c) \cdot \left.\left(2t + \frac{4}{t^2}\right)\right|_{t=2}$

$= 5c \cdot 4 + (-8c) \cdot 5 = -20c \Rightarrow c = -\frac{1}{20}$

Hence, $\vec{\nabla} f(1,-1) = -\frac{1}{4}\vec{i} + \frac{2}{5}\vec{j}$

3b. Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^5}{(x^2 + y^4)(x^4 + y^2)}$.

$x^4 \geq 0, y^2 \geq 0 \Rightarrow 0 \leq y^2 \leq x^4 + y^2 \Rightarrow 0 \leq \frac{y^2}{x^4+y^2} \leq 1$ for $(x,y) \neq (0,0)$.

$\Rightarrow 0 \leq \left| \frac{xy^5}{(x^2+y^4)\cdot(x^4+y^2)} \right| \leq \left| \frac{xy^3}{x^2+y^4} \right|$ for $(x,y) \neq (0,0)$

Since $\frac{1}{2} + \frac{3}{4} = \frac{5}{4} > 1$, by Sertöz Theorem, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^4} = 0$.

Hence, by Squeeze Theorem, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^5}{(x^2+y^4)(x^4+y^2)} = 0$

4. Consider the function $f(x,y) = x^2y(x^2+y^2-1)$.

a. Find all critical points of f. [Do not classify them!]

$$\left.\begin{aligned} f_x &= 2xy(x^2+y^2-1) + x^2y \cdot 2x = 0 \\ f_y &= x^2(x^2+y^2-1) + x^2y \cdot 2y = 0 \end{aligned}\right\} \circledast$$

If $x=0$, then both equations in $\circledast$ are satisfied.

Therefore, every point $(0,y)$ on the y-axis is a critical point.

Suppose $x \neq 0$. Then:

$\circledast \iff y(2x^2+y^2-1)=0$ and $x^2+3y^2-1=0$

$\iff (y=0$ or $2x^2+y^2=1)$ and $x^2+3y^2=1$

$\iff (y=0$ and $x^2+3y^2=1)$ or $(2x^2+y^2=1$ and $x^2+3y^2=1)$

$\iff (y=0$ and $x^2=1)$ or $(x^2=\frac{2}{5}$ and $y^2=\frac{1}{5})$

Hence the critical points of f are:

$(1,0)$, $(-1,0)$, $\left(\sqrt{\tfrac{2}{5}}, \sqrt{\tfrac{1}{5}}\right)$, $\left(\sqrt{\tfrac{2}{5}}, -\sqrt{\tfrac{1}{5}}\right)$, $\left(-\sqrt{\tfrac{2}{5}}, \sqrt{\tfrac{1}{5}}\right)$, $\left(-\sqrt{\tfrac{2}{5}}, -\sqrt{\tfrac{1}{5}}\right)$,

and $(0,y)$ for all y.

b. Choose one of the critical points of f that lies on the y-axis by filling in the box:

$$(x,y) = (0, \boxed{2})$$

Determine whether this point is a local maximum, local minimum, or a saddle point without using the 2nd Derivative Test.

If $y>1$, then $x^2 \geq 0$, $y \geq 0$, and $x^2+y^2-1 \geq y^2-1 \geq 0$.

Hence, if $y>1$, then $f(x,y) = x^2y(x^2+y^2-1) \geq 0 = f(0,2)$.

Therefore, f has a local minimum at $(0,2)$.