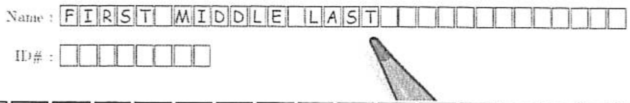
Math 102
Final Exam
May 20, 2019
9:00 - 11:00



Name :

ID# :

The comprehensive list of *Math 102 Exam Rules* **in full detail is available on Moodle.** The following are only a few reminders:

- The exam contains 5 questions, only 4 of which will be graded.

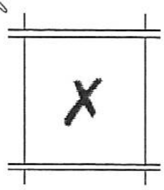
☞ `Indicate the question you do not want to be graded by crossing out its box.` ☜

- Read the questions carefully.
- **Solutions, not answers, get points.** Show all your work in well-organized mathematical sentences and explain your reasoning fully.
- **What can not be read will not be read.** Write clearly and cleanly.
- Simplify your answers as far as possible.
- Calculators and dictionaries are not allowed.
- **Turn off and leave your mobile phones with the exam proctor before the exam starts.**
- *This exam is being recorded. It is in your best interest not to give the slightest impression of doing anything improper, against the exam rules or the general rules of academic honesty.*

Do not write below this line except putting a ✗ *in one of the boxes.*

Q1	Q2	Q3	Q4	Q5	TOTAL*
25	25	25	25	25	100

* Only 4 questions will be graded. Indicate the one you do **not** want to be graded by putting a ✗ in the corresponding box, like this. The first marked question (or, if all unmarked, **Q5**) will not be graded. ☜



1. Consider the function $f(x,y,z)=\dfrac{x}{y}-\dfrac{y}{z}$ and the point $P_0(3,1,1)$.

a. Compute $\nabla f(P_0)$.

$$\vec{\nabla} f = f_x\,\vec{i}+f_y\,\vec{j}+f_z\,\vec{k} = \frac{1}{y}\vec{i}-\left(\frac{x}{y^2}+\frac{1}{z}\right)\vec{j}+\frac{y}{z^2}\vec{k}$$

$$\Rightarrow \vec{\nabla} f(P_0)=\vec{i}-4\vec{j}+\vec{k}$$

b. Is there a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(P_0)=5$? If Yes, find one. If No, prove that it does not exist.

No, because $D_{\vec{u}}f(P_0)$ can be at most $|\vec{\nabla} f(P_0)|=\sqrt{1^2+(-4)^2+1^2}=\sqrt{18}$ and $\sqrt{18}<5$.

c. Is there a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(P_0)=3$? If Yes, find one. If No, prove that it does not exist.

Yes, because if $\vec{u}=\dfrac{2\vec{i}-2\vec{j}-\vec{k}}{3}$, then

$$D_{\vec{u}}f(P_0)=\vec{\nabla} f(P_0)\cdot\vec{u}=\frac{1\cdot 2+(-4)\cdot(-2)+1\cdot(-1)}{3}=3.$$

d. Let S be the set of all points $P(x,y,z)$ where f increases fastest in the direction of the vector $\mathbf{A}=2\mathbf{i}+\mathbf{j}+2\mathbf{k}$. Show that S is a subset of the union $L_1\cup L_2$ of two lines L_1 and L_2, and find parametric equations of these lines.

$$\Leftrightarrow \vec{\nabla} f = c\cdot\vec{A} \text{ for some } c>0 \Leftrightarrow \frac{1/y}{2}=\frac{-(x/y^2+1/z)}{1}=\frac{y/z^2}{2} \text{ and } y>0$$

$$\Leftrightarrow z^2=y^2 \text{ and } \frac{1}{y}=-2\cdot\left(\frac{x}{y^2}+\frac{1}{z}\right) \text{ and } y>0$$

$$\Leftrightarrow \left(z=y \text{ and } x=-\frac{3}{2}y \text{ and } y>0\right)$$

$$\text{or } \left(z=-y \text{ and } x=\frac{1}{2}y \text{ and } y>0\right)$$

Hence, $S\subset L_1\cup L_2$ where $L_1:\ x=-\frac{3}{2}t,\ y=t,\ z=t;\ -\infty<t<\infty$

and $L_2:\ x=\frac{1}{2}t,\ y=t,\ z=-t;\ -\infty<t<\infty$.

2. Evaluate the following integrals.

a. $\iint_D x\,dA$ where $D = \{(x,y) : 1 \le x+y \le 2,\ x \ge 0 \text{ and } y \ge 0\}$

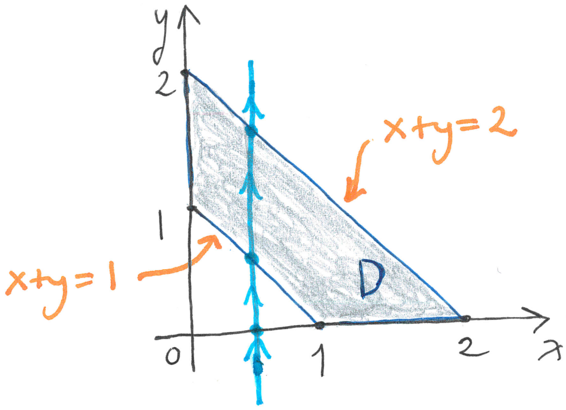
$$\iint_D x\,dA = \int_0^2\int_0^{2-x} x\,dy\,dx - \int_0^1\int_0^{1-x} x\,dy\,dx$$

$$= \int_0^2 \Big[xy\Big]_{y=0}^{y=2-x} dx - \int_0^1 \Big[xy\Big]_{y=0}^{y=1-x} dx$$

$$= \int_0^2 (2x - x^2)\,dx - \int_0^1 (x - x^2)\,dx$$

$$= \left[x^2 - \frac{1}{3}x^3\right]_0^2 - \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{4}{3} - \frac{1}{6} = \frac{7}{6}$$



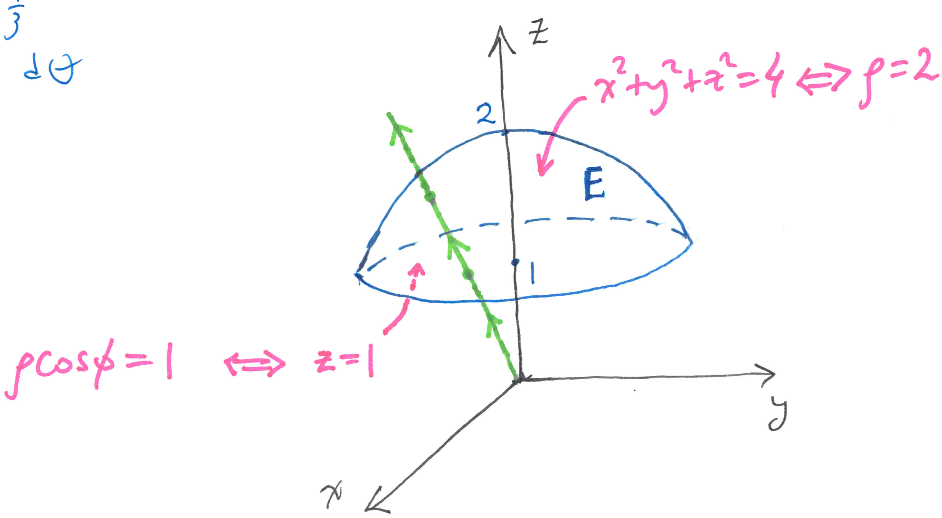
b. $\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$ where $E = \{(x,y,z) : x^2+y^2+z^2 \le 4 \text{ and } z \ge 1\}$

$$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \frac{1}{\rho}\cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{1}{2}\rho^2\right]_{\rho=\sec\phi}^{\rho=2} \sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left(2\sin\phi - \frac{1}{2}\tan\phi\sec\phi\right) d\phi\,d\theta$$

$$= \int_0^{2\pi} \left[-2\cos\phi - \frac{1}{2}\sec\phi\right]_{\phi=0}^{\phi=\frac{\pi}{3}} d\theta$$

$$= 2\pi\cdot\left(-1 - 1 + 2 + \frac{1}{2}\right) = \pi$$



3. In each of the following, if the given statement is true for all sequences $\{a_n\}_{n=1}^{\infty}$, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample. No explanation is required.

a. If $\{a_n\}_{n=1}^{\infty}$ is increasing, then $\lim\limits_{n\to\infty} a_n = \infty$.

☐ TRUE

☒ FALSE, because it does not hold for $a_n = \boxed{-\frac{1}{n}}$ for $n \geq 1$

b. If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

☐ TRUE

☒ FALSE, because it does not hold for $a_n = \boxed{\frac{1}{n}}$ for $n \geq 1$

c. If $\sum\limits_{n=1}^{\infty} a_n$ converges conditionally, then $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ diverges.

☐ TRUE

☒ FALSE, because it does not hold for $a_n = \boxed{\frac{\sin(n\pi/2)}{n}}$ for $n \geq 1$

d. If $\sum\limits_{n=1}^{\infty} a_n^2$ converges, then $\sum\limits_{n=1}^{\infty} a_n^3$ converges.

☒ TRUE

☐ FALSE, because it does not hold for $a_n = \boxed{}$ for $n \geq 1$

e. If $\sum\limits_{n=1}^{\infty} a_n^2$ diverges, then $\sum\limits_{n=1}^{\infty} a_n^3$ diverges.

☐ TRUE

☒ FALSE, because it does not hold for $a_n = \boxed{\frac{1}{\sqrt{n}}}$ for $n \geq 1$

4. Determine whether each of the following series converges or diverges.

a. $\sum_{n=1}^{\infty} n^2(\arctan(n+1) - \arctan(n))$

$\frac{\pi}{2} - \frac{\pi}{2} = 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2(\arctan(n+1) - \arctan(n)) = \lim_{x\to\infty} \frac{\arctan(x+1) - \arctan(x)}{\frac{1}{x^2}}$$

$\frac{1}{x^2} \to 0$

L'H

$$= \lim_{x\to\infty} \frac{\frac{1}{(x+1)^2+1} - \frac{1}{x^2+1}}{-\frac{2}{x^3}} = \lim_{x\to\infty} \frac{(2x+1)x^3}{2(x^2+2x+2)(x^2+1)} = 1 \neq 0.$$

By n^{th} Term Test, $\sum_{n=1}^{\infty} n^2(\arctan(n+1) - \arctan(n))$ diverges.

b. $\sum_{n=1}^{\infty} n(\arctan(n+1) - \arctan(n))$

$$c = \lim_{n\to\infty} \frac{n(\arctan(n+1) - \arctan(n))}{1/n} = \lim_{n\to\infty} n^2(\arctan(n+1) - \arctan(n)) = 1$$

Since $c > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) diverges,

by Limit Comparison Test, $\sum_{n=1}^{\infty} n(\arctan(n+1) - \arctan(n))$ diverges.

c. $\sum_{n=1}^{\infty} (\arctan(n+1) - \arctan(n))$

$$c = \lim_{n\to\infty} \frac{\arctan(n+1) - \arctan(n)}{1/n^2} = \lim_{n\to\infty} n^2(\arctan(n+1) - \arctan(n)) = 1$$

Since $c < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (p-series with $p = 2 > 1$) converges,

by Limit Comparison Test, $\sum_{n=1}^{\infty} (\arctan(n+1) - \arctan(n))$ converges.

5. Suppose that the function $f(x)$ defined on the interval $(-R, R)$ by a power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

which has radius of convergence $R > 0$ satisfies $f(0) = 1$ and $f'(x) = f(x/2)$ for all x in $(-R, R)$.

a. Express the coefficients of the following series in terms of $c_0, c_1, c_2, \ldots, c_n, \ldots$ by filling in the boxes. No explanation is required.

$$f'(x) = \boxed{c_1} + \boxed{2c_2}\, x + \boxed{3c_3}\, x^2 + \cdots + \boxed{(n+1)c_{n+1}}\, x^n + \cdots$$

$$f(x/2) = \boxed{c_0} + \boxed{\tfrac{1}{2}c_1}\, x + \boxed{\tfrac{1}{4}c_2}\, x^2 + \cdots + \boxed{\tfrac{1}{2^n}c_n}\, x^n + \cdots$$

b. Give a recurrence relation for $\{c_n\}_{n=0}^{\infty}$ by filling in the boxes. No explanation is required.

$$c_0 = \boxed{1} \quad \text{and} \quad c_{n+1} = \boxed{\frac{1}{2^n \cdot (n+1)}} \cdot c_n \quad \text{for } n \geq 0$$

c. Find R.

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1}{2^n \cdot (n+1)} = 0 \Rightarrow R = \infty$$

d. Determine whether $f(-2)$ is positive or negative.

$$f(-2) = \sum_{n=0}^{\infty} c_n \cdot (-2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 2^n c_n = \sum_{n=0}^{\infty} (-1)^n b_n \quad \text{where } b_n = 2^n c_n > 0 \text{ for } n \geq 0.$$

By Part c, $\lim_{n\to\infty} b_n = 0$

By Part b, $b_{n+1} = \dfrac{1}{2^{n-1} \cdot (n+1)} b_n \leq b_n$ for $n \geq 1$ as $2^{n-1} \cdot (n+1) \geq 1 \cdot 1 = 1$ for $n \geq 1$

Therefore, by AST:

$$f(-2) \leq \sum_{n=0}^{4} (-1)^n b_n = 1 - 2 + 1 - \frac{1}{6} + \frac{1}{96} = -\frac{5}{32} < 0$$

$\Rightarrow f(-2)$ is negative.

$$x = r\cos\theta \qquad r = \rho\sin\phi \qquad x = \rho\,\sin\phi\,\cos\theta$$
$$y = r\sin\theta \qquad \theta = \theta \qquad y = \rho\,\sin\phi\,\sin\theta$$
$$z = z \qquad z = \rho\cos\phi \qquad z = \rho\,\cos\phi$$

$$dA = dy\,dx = r\,dr\,d\theta$$

$$dV = dz\,dy\,dx = r\,dz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\lim_{n\to\infty} x^n = 0 \text{ for all constant } |x| < 1$$
$$\lim_{n\to\infty} x^{1/n} = 1 \text{ for all constant } x > 0$$
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all constant } x$$
$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \text{ for all constant } x$$
$$\lim_{n\to\infty} \frac{(\ln n)^a}{n^b} = 0 \text{ for all constants } a \text{ and } b > 0$$
$$\lim_{n\to\infty} n^{1/n} = 1$$
$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$
$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$$
$$\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \leq 1$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } |x| \leq 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$