1a. Find parametric equations of the line of intersection L of the planes:

$$\mathscr{M} : x + 3y + 5z = -1 \qquad \text{and} \qquad \mathscr{N} : 3x + 2y + z = 4$$

$$\Downarrow \qquad\qquad \Downarrow$$

Let $x = t$. Then $3y + 5z = -t - 1$ ① and $2y + z = -3t + 4$ ②

$5 \times ② - ①$ gives $7y = -14t + 21 \Rightarrow y = -2t + 3 \xRightarrow{②} z = t - 2$

Hence, $L: \; x = t, \; y = -2t + 3, \; z = t - 2; \quad -\infty < t < \infty$.

1b. Suppose that $\mathbf{r}(t) = \overrightarrow{OP}(t)$ is a parametric curve such that the point $P(t)$ lies on the plane with equation

$$\mathscr{P}(t) : e^t x + e^{2t} y + e^{3t} z = 1$$

for each t. Show that if $\left.\dfrac{d}{dt}\mathbf{r}(t)\right|_{t=0} = \mathbf{0}$, then the point $P(0)$ belongs to the line L in **Part 1a**.

Let $\vec{r}(t) = \vec{OP}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$.

$e^t x(t) + e^{2t} y(t) + e^{3t} z(t) = 1$ for all t $\xRightarrow{t=0}$ $x(0) + y(0) + z(0) = 1$ ③

$\Downarrow d/dt$

$e^t x(t) + e^t x'(t) + 2e^{2t} y(t) + e^{2t} y'(t) + 3e^{3t} z(t) + e^{3t} z'(t) = 0$

$\Downarrow t=0$

④ $x(0) + 2y(0) + 3z(0) = 0$ as $x'(0) = y'(0) = z'(0) = 0$

$4 \times ③ - ④$ gives $3x(0) + 2y(0) + z(0) = 4$

$2 \times ④ - ③$ gives $x(0) + 3y(0) + 5z(0) = -1$

$\Big\} \Rightarrow (x(0), y(0), z(0))$ lies on L.

2a. Make each of the following sentences into a true statement by choosing one of the possible completions. Indicate your choice by putting a ✗ in the corresponding box. No explanation is required.

① $\lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2}$ ☐ exists ☒ does not exist

② $\lim_{(x,y)\to(0,0)} \frac{y(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2}$ ☐ exists ☒ does not exist

③ $\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2}$ ☒ exists ☐ does not exist

2b. Now prove two of your statements in **Part 2a**. Write the number of the statement you are proving inside the circle.

- I will prove the statement 1 here.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the x-axis}}} \frac{x(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2} = \lim_{x\to0} \frac{x^3}{2x^4} = \frac{1}{2}\lim_{x\to0}\frac{1}{x} \text{ does not exist}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2} \text{ does not exist by the 1-Path Test.}$$

- I will prove the statement 2 here.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the x-axis}}} \frac{y(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2} = \lim_{x\to0} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along the parabola } y=x^2}} \frac{y(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2} = \lim_{x\to0} \frac{x^2(x^2-x^4)}{4x^4} = \frac{1}{4}\lim_{x\to0}(1-x^2) = \frac{1}{4}$$

$$0 \neq \frac{1}{4} \Rightarrow \lim_{(x,y)\to(0,0)} \frac{y(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2} \text{ does not exist by the 2-Path Test.}$$

• I will prove the statement 3 here.

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{(y-x^2)^2+(y+x^2)^2} = \frac{1}{2}\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} - \frac{1}{2}\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^2} = 0-0=0,$$

where the first limit is zero since $\frac{3}{4}+\frac{1}{2}=\frac{5}{4}>1$, and the second limit is zero since $\frac{1}{4}+\frac{3}{2}=\frac{7}{4}>1$.

3. The *combustion equation*

$$u_{xx} + u_{yy} = -e^u$$

arises in the study of self-propagating exothermic oxidative chemical reactions in thermochemistry. Find all possible values of the triple (a, b, c) of constants for which the function

$$u(x, y) = a \ln(bx^2 + by^2 + c)$$

satisfies the combustion equation for all (x, y) with $x^2 + y^2 < 1$ as well as the condition $u(x, y) = 0$ for all (x, y) with $x^2 + y^2 = 1$.

$$u_x = a \cdot \frac{1}{bx^2+by^2+c} \cdot 2bx$$

$$u_{xx} = a \cdot \frac{-1}{(bx^2+by^2+c)^2} \cdot (2bx)^2 + a \cdot \frac{1}{bx^2+by^2+c} \cdot 2b$$

Similarly:

$$u_{yy} = a \cdot \frac{-1}{(bx^2+by^2+c)^2} \cdot (2by)^2 + a \cdot \frac{1}{bx^2+by^2+c} \cdot 2b$$

$$u_{xx}+u_{yy} = \frac{-4ab^2(x^2+y^2) + 4ab(bx^2+by^2+c)}{(bx^2+by^2+c)^2} = \frac{4abc}{(bx^2+by^2+c)^2}$$

$$e^u = \exp(a \ln(bx^2+by^2+c)) = (bx^2+by^2+c)^a$$

Hence: $u_{xx}+u_{yy} = -e^u$ for all $x^2+y^2<1 \iff a=-2$ and $4abc=-1 \iff a=-2$ and $bc=\frac{1}{8}$

As $a \neq 0$, $u(x,y)=0$ for all $x^2+y^2=1 \iff a \ln(b+c)=0 \iff b+c=1$

$$\left.\begin{array}{l} bc=\frac{1}{8} \Rightarrow c=\frac{1}{8b} \\ b+c=1 \end{array}\right\} \Rightarrow b+\frac{1}{8b}=1 \Rightarrow b^2-b+\frac{1}{8}=0 \Rightarrow b=\frac{1}{2}\left(1\pm\frac{1}{\sqrt{2}}\right)$$

$$\Downarrow$$

$$c=\frac{1}{2}\left(1\mp\frac{1}{\sqrt{2}}\right)$$

Hence, $(a,b,c)=\left(-2, \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right), \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\right)$ and $\left(-2, \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right), \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)\right)$

are the only triples satisfying the conditions.

4a. Find all pairs (u, v) of real numbers satisfying both of the equations $uv = 6$ and $u^2 - v^2 = 5$.

$$\left.\begin{array}{l} uv = 6 \Rightarrow \quad v = \dfrac{6}{u} \\ u^2 - v^2 = 5 \end{array}\right\} \Rightarrow u^2 - \frac{36}{u^2} = 5 \Rightarrow (u^2)^2 - 5u^2 - 36 = 0$$

$$\Rightarrow (u^2 - 9)(u^2 + 4) = 0 \Rightarrow u^2 = 9 \quad \text{or} \quad u^2 = -4 \ \otimes$$

$$\Downarrow$$

$$u = 3 \quad \text{or} \quad u = -3$$

$$\Downarrow \qquad \qquad \Downarrow$$

$$v = 2 \qquad \qquad v = -2$$

$(u, v) = (3, 2)$ and $(-3, -2)$ are the only solutions.

4b. Find $\left.\dfrac{\partial f}{\partial x}\right|_{(x,y)=(6,5)}$ if $f(x, y)$ is a differentiable function satisfying

$$f(uv, u^2 - v^2) = u^3 + v^3 \qquad (*)$$

for all $u > 0$ and $v > 0$.

$$\left.\begin{array}{l} (*) \xrightarrow{\frac{\partial}{\partial u}} f_1(uv, u^2-v^2) \cdot v + f_2(uv, u^2-v^2) \cdot 2u = 3u^2 \\ (*) \xrightarrow{\frac{\partial}{\partial v}} f_1(uv, u^2-v^2) \cdot u + f_2(uv, u^2-v^2) \cdot (-2v) = 3v^2 \end{array}\right\}$$

$$\xrightarrow{(u,v)=(3,2)} \begin{cases} 2 f_1(6,5) + 6 f_2(6,5) = 27 & (1) \\ 3 f_1(6,5) - 4 f_2(6,5) = 12 & (2) \end{cases}$$

$2 \times (1) + 3 \times (2)$ gives $13 f_1(6,5) = 90 \Rightarrow \left.\dfrac{\partial f}{\partial x}\right|_{(x,y)=(6,5)} = \dfrac{90}{13}$