Bilkent University

Quiz # 08
Math 102 Section 03 Calculus II
16 April 2020
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Solution Key

Q-1) Consider the sequence defined by

$$a_0 = 6, \quad a_n = \sqrt{6 + a_{n-1}},\ n \geq 1.$$

(i) Show that the sequence is decreasing.

(ii) Show that the sequence is bounded.

(iii) Explain why the sequence converges or diverges.

(iv) Assuming that the sequence converges, find $\lim_{n\to\infty} a_n$.

Grading: (i) 4 points, (ii) 1 point, (iii) 1 point, (iv) 4 points

Solution:

(i) We do this by induction.
For $n = 1$ we have $a_1 = \sqrt{6 + a_0} = 2\sqrt{3} < a_0$.
We assume that $a_n < a_{n-1}$. Then $a_{n+1} = \sqrt{6 + a_n} < \sqrt{6 + a_{n-1}} = a_n$.
Hence by induction, $a_n < a_{n-1}$ for all $n \geq 1$, and the sequence is decreasing.

(ii) We clearly have $0 < a_n < 6$, so the sequence is bounded.

(iii) Since every monotone, bounded sequence converges, our sequence being monotone by (i), and bounded by (ii), converges.

(iv) Let $\lim_{n\to\infty} a_n = L$. Taking the limit of both sides of $a_n = \sqrt{6 + a_{n-1}}$ as $n \to \infty$, we get $L = \sqrt{6 + L}$. This gives the quadratic equation $L^2 - L - 6 = 0$, whose roots are -2 and 3. Since $a_n > 0$, we must have $L \geq 0$. Hence $L = 3$.