Bilkent University

Quiz # 09
Math 102 Section 03 Calculus II
30 April 2020
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Solution Key

Q-1)

(i) Find all real numbers $a \geq 0$ such that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a}$ converges.

(ii) Find all real numbers $a \geq 0$ such that the series $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^a}{n^3}$ converges.

Solution:

(i) We use the Integral Test. When $a = 1$ we have

$$\int_2^{\infty} \frac{dx}{x \ln x} = \left(\ln \ln x \Big|_2^{\infty} \right) = \infty,$$

hence the series diverges when $a = 1$.
When $a \geq 0$ but $a \neq 1$ we have

$$\int_2^{\infty} \frac{dx}{x(\ln x)^a} = \left(\frac{(\ln x)^{1-a}}{1-a} \Bigg|_2^{\infty} \right) = \begin{cases} \infty & \text{if } 0 \leq a < 1 \\ \text{finite} & \text{if } a > 1 \end{cases}.$$

We then conclude that the series converges only when $a > 1$.

(ii) We limit-compare with $\sum 1/n^2$.

$$\lim_{x\to\infty} \frac{(\ln x)^a/x^3}{1/x^2} = \lim_{x\to\infty} \frac{(\ln x)^a}{x} = 0, \quad \text{By L'Hospital, for all } a \geq 0,$$

which means

$$\frac{(\ln x)^a}{x^3} < \frac{1}{x^2}, \quad \text{for all large } x.$$

Hence

$$\sum_{n=2}^{\infty} \frac{(\ln n)^a}{x^3} \quad \text{converges for all } a \geq 0.$$