Bilkent University

Quiz # 10
Math 102 Section 03 Calculus II
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Solution Key

Q-1) Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^{2020}}{(2020n)!}\, x^n$.

(i) Find the radius of convergence R of this series.

(ii) Find if the series converges when $x = R$, and when $x = -R$.

Grading: (i) 8 points, (ii) 1+1 points

Solution:

(i) Let $a_n = \dfrac{(n!)^k}{(kn)!}\, x^n$, where $k = 2020$. We use the Ratio test for absolute convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{[(n+1)!]^k (kn)!}{[k(n+1)]!\,(n!)^k}\,|x| \\
&= \lim_{n\to\infty} \frac{(n+1)^k}{(kn+1)(kn+2)\cdots(kn+k)}\,|x| \\
&= \lim_{n\to\infty}\left(\frac{n+1}{kn+1}\right)\lim_{n\to\infty}\left(\frac{n+1}{kn+2}\right)\cdots\lim_{n\to\infty}\left(\frac{n+1}{kn+k}\right)|x| \qquad (*) \\
&= \left(\frac{1}{k}\right)\left(\frac{1}{k}\right)\cdots\left(\frac{1}{k}\right)|x| \\
&= \left(\frac{1}{k}\right)^k |x| < 1,
\end{aligned}$$

gives $|x| < k^k$ for absolute convergence. Hence $R = k^k = 2020^{2020}$.

(ii) When you put $x = k^k$ in $(*)$ in the above equation, without the limit operation, you get

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{kn+k}{kn+1}\,\frac{kn+k}{kn+2}\cdots\frac{kn+k}{kn+k} > 1,$$

which shows that $|a_{n+1}| > |a_n|$ when $x = \pm k^k$. Hence the general term does not go to zero and the series diverges at the end points.