Bilkent University

Quiz # 04
Math 102 - Calculus II - Section 03
3 March 2022 Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Prove the following magical formula

$$\frac{1}{2} = 1 - \frac{\pi^2}{2!\,3^2} + \frac{\pi^4}{4!\,3^4} - \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!\,3^{2n}} + \cdots .$$

Hint: First find the Taylor expansion of $\sin x$ at $x = \pi/2$ and then see what you can do from there.

Grading: 10 points

Solutions:

Let $f(x) = \sin x$. Then we have

$$\begin{aligned} f(x) &= \sin x & f(\pi/2) &= 1 \\ f'(x) &= \cos x & f'(\pi/2) &= 0 \\ f''(x) &= -\sin x & f''(\pi/2) &= -1 \\ f'''(x) &= -\cos x & f'''(\pi/2) &= 0 \end{aligned}$$

and the pattern repeats. Hence we have

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (x - \frac{\pi}{2})^{2n}$$

where we can take $a_n = (-1)^n \frac{1}{(2n)!} (x - \frac{\pi}{2})^{2n}$ and

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n+2)!}{(2n)!} |x - \frac{\pi}{2}| = (2n+1)(2n+2)|x - \frac{\pi}{2}| \to 0 < 1 \text{ as } n \to \infty \text{ for all } x.$$

Hence the series converges for all x. In particular substituting $x = \pi/6$ we get the above magical formula.