1. Let $f(x,y) = 2\cos(x+y) - 3 + e^{-(x^2+y^2)}$.

a. Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)}$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)} = \lim_{x\to 0} \frac{x\cdot 0}{f(x,0)} = \lim_{x\to 0} \frac{0}{2\cos x - 3 + e^{-x^2}} = \lim_{x\to 0} 0 = 0$$

along the x-axis

$$\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)} = \lim_{x\to 0} \frac{x\cdot(-x)}{f(x,-x)} = \lim_{x\to 0} \frac{-x^2}{-1+e^{-2x^2}} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-2x}{-4xe^{-2x^2}} = \frac{1}{2}$$

along the line $y=-x$

Since $0 \neq \frac{1}{2}$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{f(x,y)}$ does not exist by 2-Path Test.

b. $(0,0)$ is a critical point of f. [You do not need to verify this.] Determine whether f has a local maximum, a local minimum, a saddle point, or something else at $(0,0)$.

$$\left\{\begin{array}{l} \cos(x+y) \leq 1 \text{ for all } (x,y) \\ -(x^2+y^2) \leq 0 \Rightarrow e^{-(x^2+y^2)} \leq 1 \text{ for all } (x,y) \end{array}\right.$$

$\Rightarrow f(x,y) = 2\cos(x+y) - 3 + e^{-(x^2+y^2)} \leq 2 - 3 + 1 = 0 = f(0,0)$ for all (x,y)

$\Rightarrow f$ has its absolute maximum value at $(0,0)$

$\Rightarrow f$ has a local maximum value at $(0,0)$

2. Let $f(x,y,z) = x^3 - xy^2z + \dfrac{8}{z^2}$ and $P_0(1,-1,2)$.

a. Compute $\nabla f(P_0)$.

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (3x^2 - y^2 z)\vec{i} - 2xyz\vec{j} + \left(-xy^2 - \frac{16}{z^3}\right)\vec{k}$$

$$\Rightarrow \vec{\nabla} f(P_0) = \vec{i} + 4\vec{j} - 3\vec{k}$$

b. Is there a unit vector **u** such that the rate of change of f at P_0 in the direction of **u** is 0? If YES, find one; if NO, explain why it does not exist.

YES. For $\vec{u} = \frac{3}{5}\vec{j} + \frac{4}{5}\vec{k}$,

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = (\vec{i} + 4\vec{j} - 3\vec{k}) \cdot \left(\frac{3}{5}\vec{j} + \frac{4}{5}\vec{k}\right) = 4 \cdot \frac{3}{5} - 3 \cdot \frac{4}{5} = 0$$

c. Is there a unit vector **u** such that the rate of change of f at P_0 in the direction of **u** is 5? If YES, find one; if NO, explain why it does not exist.

YES. For $\vec{u} = \frac{4}{5}\vec{j} - \frac{3}{5}\vec{k}$,

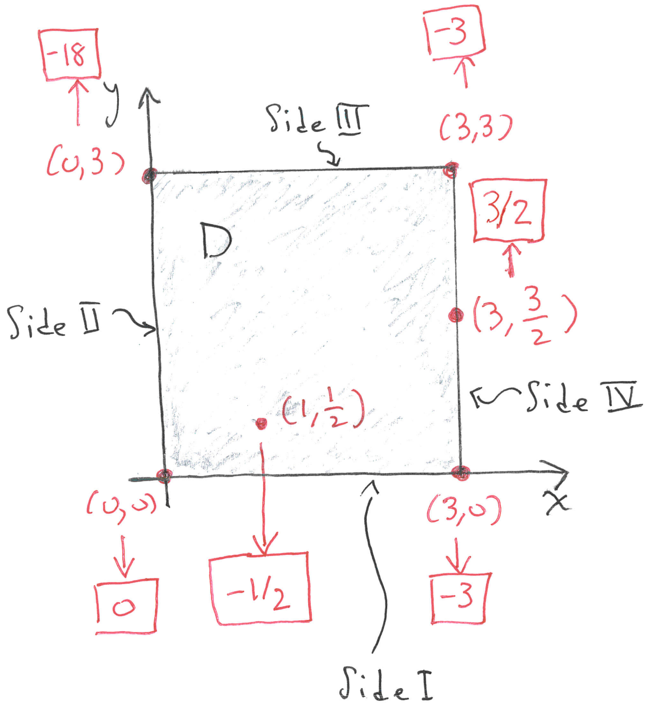
$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = (\vec{i} + 4\vec{j} - 3\vec{k}) \cdot \left(\frac{4}{5}\vec{j} - \frac{3}{5}\vec{k}\right) = 4 \cdot \frac{4}{5} - 3 \cdot \left(-\frac{3}{5}\right) = 5$$

d. Is there a unit vector **u** such that the rate of change of f at P_0 in the direction of **u** is -7? If YES, find one; if NO, explain why it does not exist.

No.

No such $\vec{u}$ exists as $-|\vec{\nabla} f(P_0)| = -|\vec{i} + 4\vec{j} - 3\vec{k}| = -\sqrt{26} > -7$.

3. Find the absolute maximum and minimum values of the function $f(x, y) = 2xy - x - 2y^2$ on the square $D = \{(x, y) : 0 \le x \le 3 \text{ and } 0 \le y \le 3\}$.



Interior of D:

$f_x = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$

$f_y = 2x - 4y = 0 \Rightarrow x = 1$

$\Rightarrow (x, y) = (1, \frac{1}{2})$

Boundary of D:

Side I: $0 \le x \le 3$ and $y = 0$

$f(x, 0) = -x$ for $0 \le x \le 3$

Critical points: $\frac{d}{dx} f(x, 0) = -1 = 0 \Rightarrow$ no solution

Endpoints: $x = 0, x = 3 \Rightarrow (x, y) = (0, 0), (3, 0)$

Side II: $0 \le y \le 3$ and $x = 0$

$f(0, y) = -2y^2$ for $0 \le y \le 3$

Critical points: $\frac{d}{dy} f(0, y) = -4y = 0 \Rightarrow y = 0$

Endpoints: $y = 0, y = 3 \Rightarrow (x, y) = (0, 0), (0, 3)$

Side III: $0 \le x \le 3$ and $y = 3$

$f(x, 3) = 5x - 18$ for $0 \le x \le 3$

Critical points: $\frac{d}{dx} f(x, 3) = 5 = 0 \Rightarrow$ no solution

Endpoints: $x = 0, x = 3 \Rightarrow (x, y) = (0, 3), (3, 3)$

Side IV: $0 \le y \le 3$ and $x = 3$

$f(3, y) = 6y - 3 - 2y^2$ for $0 \le y \le 3$

Critical points: $\frac{d}{dy} f(3, y) = 6 - 4y = 0 \Rightarrow y = \frac{3}{2} \Rightarrow (x, y) = (3, \frac{3}{2})$

Endpoints: $y = 0, y = 3 \Rightarrow (x, y) = (3, 0), (3, 3)$

Absolute maximum value is $\frac{3}{2}$, absolute minimum value is -18.

4. Evaluate the following integrals.

a. $\displaystyle\iint_D x^2\cos(\pi xy)\,dA$ where $D=\{(x,y): 0\le y\le x \text{ and } 0\le x\le 1\}$

$$\iint_D x^2\cos(\pi xy)\,dA=\int_0^1\int_0^x x^2\cos(\pi xy)\,dy\,dx$$

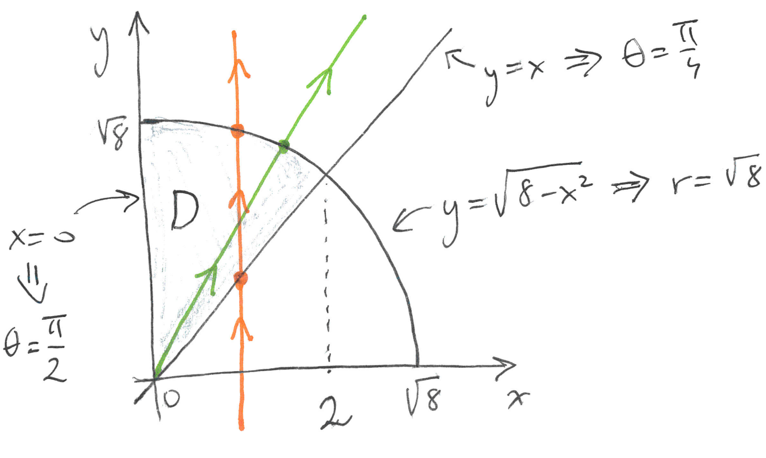
$$=\int_0^1\left[\frac{1}{\pi}x\sin(\pi xy)\right]_{y=0}^{y=x}dx=\int_0^1\frac{1}{\pi}x\sin(\pi x^2)\,dx$$

$$=\left[-\frac{1}{2\pi^2}\cos(\pi x^2)\right]_0^1=-\frac{1}{2\pi^2}\cdot(\cos\pi-\cos 0)=\frac{1}{\pi^2}$$

b. $\displaystyle\int_0^2\int_x^{\sqrt{8-x^2}}\arctan(y/x)\,dy\,dx=\iint_D\arctan(y/x)\,dA=\int_{\pi/4}^{\pi/2}\int_0^{\sqrt8}\theta\cdot r\,dr\,d\theta$

$$=\int_{\pi/4}^{\pi/2}\theta\cdot\frac12 r^2\Big]_{r=0}^{r=\sqrt8}d\theta=4\int_{\pi/4}^{\pi/2}\theta\,d\theta=2\theta^2\Big]_{\pi/4}^{\pi/2}$$

$$=2\cdot\left(\frac{\pi^2}{4}-\frac{\pi^2}{16}\right)=\frac{3\pi^2}{8}$$



5. Let $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 2z \text{ and } z \leq 1\}$.

a. Fill in the boxes so that the following equality holds for all continuous functions f where (r, θ, z) are the cylindrical coordinates.

$$\iiint_E f(x,y,z)\,dV = \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{1-\sqrt{1-r^2}}}^{\boxed{1}} f(r\cos\theta, r\sin\theta, z)\,\boxed{r}\,dz\,dr\,d\theta$$

b. Fill in the boxes so that the following equality holds for all continuous functions f where (ρ, ϕ, θ) are the spherical coordinates.

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{\pi/4}} \int_{\boxed{0}}^{\boxed{\sec\phi}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\boxed{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta$$

$$+ \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{2\cos\phi}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\boxed{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta$$

$z = 1 \pm \sqrt{1-r^2}$

$\Uparrow$

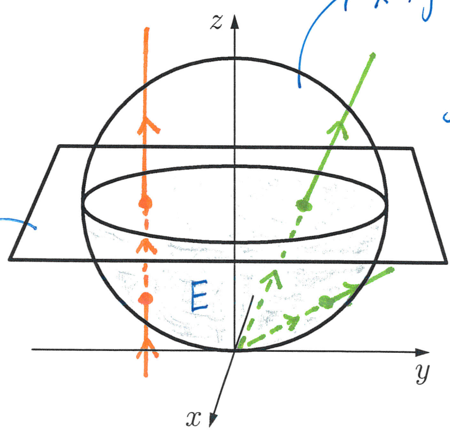
$r^2 + z^2 = 2z$

$\Uparrow$

$x^2 + y^2 + z^2 = 2z$

$\Downarrow$

$\rho = 2\cos\phi$



$z = 1$

$\Downarrow$

$\rho = \sec\phi$