Bilkent University

Quiz # 1
Math 102-Section 11
14 March 2023, Tuesday, Moodle Quiz
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Solution Key

Q-1) We have two sequences $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ and $b_n = \dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$, $n \geq 1$.

(i) Show that $\dfrac{n}{n+1} < \dfrac{n+1}{n+2}$ for all $n \geq 1$.

(ii) Show that $a_n < b_n$, $n \geq 1$

(iii) Show that $a_n < \dfrac{1}{\sqrt{2n+1}}$, $n \geq 1$.

(iv) Show that $\lim\limits_{n\to\infty} a_n = 0$.

To prove an item you can use the statements preceding it.

Show your work in detail. Correct answers without detailed explanation do not get any credit.

Grading: 1+4+4+1=10 points.

Solution:

(i) Since $n(n+2) = n^2 + 2n < n^2 + 2n + 1 = (n+1)^2$, the claimed inequality follows.

(ii) By (i) we have $\dfrac{1}{2} < \dfrac{2}{3}$, $\dfrac{3}{4} < \dfrac{4}{5}, \ldots, \dfrac{2n-1}{2n} < \dfrac{2n}{2n+1}$.

All these inequalities follow from (i).
Now multiplying side by side we get $a_n < b_n$, for all $n \geq 1$

(iii)

$$\begin{aligned} a_n^2 = a_n \cdot a_n &< a_n \cdot b_n, \quad \text{since by (ii) } a_n < b_n. \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \\ &= \frac{1}{2n+1}, \text{ for all } n \geq 1. \end{aligned}$$

This shows that $a_n < \dfrac{1}{\sqrt{2n+1}}$, for $n \geq 1$.

(iv) Since obviously $a_n > 0$, using (iii) we have $0 < a_n < \dfrac{1}{\sqrt{2n+1}}$. Now using the squeeze theorem we get $\lim\limits_{n\to\infty} a_n = 0$.