Bilkent University

Quiz # 2
Math 102-Section 09
20 March 2023, Monday, Moodle Quiz
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Solution Key

Q-1) Let $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

(i) Find all x for which this series converges absolutely.

(ii) Show that $f(x)$ satisfies the differential equation

$$y'' + y = 0.$$

(iii) A theorem on differential equations says that if $g(x)$ is a solution of the above differential equation, then

$$g(x) = A\cos x + B\sin x,$$

where A and B are some constants. Show that $f(x) = \sin x$.

Show your work in detail. Correct answers without detailed explanation do not get any credit.

Grading: 4+3+3=10 points.

Solution:

(i) Let $a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Then using the ratio test for absolute convergence we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x^2|}{(2n+2)(2n+3)} \to 0 \text{ as } n \to \infty, \text{ for all } x.$$

This shows that the series converges absolutely for all values of x.

(ii) Taking successive derivatives we have

$$\begin{aligned} f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + (-1)^{n+1}\frac{x^{2n+3}}{(2n+3)!} + \cdots \\ f'(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^{n+1}\frac{x^{2n+2}}{(2n+2)!} + \cdots \\ f''(x) &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

We see that $f''(x) + f(x) = 0$ as claimed.

(iii) By the quoted theorem we must have

$$f(x) = A\cos x + B\sin x.$$

Calculating $f(0)$ and $f'(0)$ first from the power series expansion and then from the above form we see that $A = 0$ and $B = 1$. Hence $f(x) = \sin x$.