Bilkent University

Quiz # 4
Math 102-Section 09
26 April 2023, Wednesday, Moodle Quiz
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Solution Key

Q-1) Consider the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Calculate $f_x(0,0)$ and $f_y(0,0)$.

(b) Calculate $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$.

(c) Are f_x and f_y continuous at $(0,0)$?

(d) Is f differentiable at $(0,0)$?

Show your work in detail. Correct answers without detailed explanation do not get any credit.

Grading: 2+2+2+4=10 points.

Solution:

(1-a)

$$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} x\sin\frac{1}{x^2+y^2} = 0, \text{ by the Squeeze Theorem.}$$

Similarly $f_y(0,0) = 0$.

(1-b) When $(x,y) \neq (0,0)$ we have

$$f_x(x,y) = \frac{\partial}{\partial x}\left((x^2+y^2)\sin\frac{1}{x^2+y^2}\right) = 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}.$$

Similarly

$$f_y(x,y) = \frac{\partial}{\partial y}\left((x^2+y^2)\sin\frac{1}{x^2+y^2}\right) = 2y\sin\frac{1}{x^2+y^2} - \frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}.$$

(1-c) These are not continuous at the origin. For example for the continuity of $f_x(x,y)$ at the origin we must check if

$$\lim_{(x,y)\to(0,0)} f_x(x,y) \stackrel{?}{=} f_x(0,0) = 0.$$

However these limits do not exist. For example if we approach the origin along the x-axis as $x(n) = \left(\frac{1}{2n\pi}\right)^{1/2}$, then

$$\lim_{\substack{(x,y)\to(0,0)\\ x=x(n),y=0}} f(x,y) = \lim_{n\to\infty} 2\sqrt{2n\pi} = \infty.$$

Similarly if we choose $x = 0$ and $y = \left(\frac{1}{2n\pi}\right)^{1/2}$, then

$$\lim_{\substack{(x,y)\to(0,0)\\ x=x(n),y=0}} f(x,y) = \lim_{n\to\infty} 2\sqrt{2n\pi} = \infty.$$

(1-d) We now know that $f_x(0,0) = 0$ and $f_y(0,0) = 0$. Define two error functions as

$$\epsilon_1(x,y) = \begin{cases} x \sin \frac{1}{x^2+y^2} & (x,y) \neq 0, \\ 0 & (x,y) = (0,0), \end{cases}$$

and

$$\epsilon_2(x,y) = \begin{cases} y \sin \frac{1}{x^2+y^2} & (x,y) \neq 0, \\ 0 & (x,y) = (0,0). \end{cases}$$

Now we see that

$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \epsilon_1 x + \epsilon_2 y.$$

Hense f is differentiable at the origin even though its partial derivatives are not continuous there.

Here is a graph of this function around the origin. Observe how the graph flattens out at the origin and there the plane $z = 0$ becomes tangent to the surface.