1a. Let $f(x) = \underbrace{x + \frac{x^3}{2} + \frac{x^5}{4} + \frac{x^7}{8} + \cdots + \frac{x^{2n+1}}{2^n} + \cdots}$. Find all x satisfying the equation $f(x) = -1$.

This is a geometric series with $a = x$ and $r = \frac{x^2}{2}$.

$$\Rightarrow \begin{cases} f(x) = \dfrac{x}{1-\frac{x^2}{2}} \text{ if } |r| = \left|\frac{x^2}{2}\right| < 1 & [\Leftrightarrow |x| < \sqrt{2}] \\ f(x) \text{ is undefined if } |r| = \left|\frac{x^2}{2}\right| \geq 1 & [\Leftrightarrow |x| \geq \sqrt{2}] \end{cases}$$

$$f(x) = -1 \Leftrightarrow \frac{x}{1-\frac{x^2}{2}} = -1 \Leftrightarrow x^2 - 2x - 2 = 0 \Leftrightarrow x = 1 - \sqrt{3} \text{ or } x = 1 + \sqrt{3}.$$

$1 + \sqrt{3} > 1 + 1 = 2 > \sqrt{2} \Rightarrow f(1+\sqrt{3})$ is undefined.

$0 < \sqrt{3} - 1 < 1 < \sqrt{2} \Rightarrow x = 1 - \sqrt{3}$ is the only solution.

1b. Let $A = \frac{1}{2} + \underbrace{\frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \cdots + \frac{1}{(2n+1) \cdot 2^{2n+1}} + \cdots}_{B}$. Show that $A < \frac{5}{9}$.

$$B < \underbrace{\frac{1}{3 \cdot 2^3} + \frac{1}{3 \cdot 2^5} + \cdots + \frac{1}{3 \cdot 2^{2n+1}} + \cdots} = \frac{\frac{1}{3 \cdot 2^3}}{1 - \frac{1}{2^2}} = \frac{1}{18}$$

This geometric series with $a = \frac{1}{3 \cdot 2^3}$ and $r = \frac{1}{2^2}$

converges as $|r| = \left|\frac{1}{2^2}\right| = \frac{1}{4} < 1$, and

Therefore,

$$A = \frac{1}{2} + B < \frac{1}{2} + \frac{1}{18} = \frac{10}{18} = \frac{5}{9}$$

2. Determine whether each of the following series converges or diverges.

a. $\sum_{n=0}^{\infty}(\sqrt{n+1}-\sqrt{n})$

$$S_n=\sum_{k=0}^{n}(\sqrt{k+1}-\sqrt{k})=(\sqrt{1}-\sqrt{0})+(\sqrt{2}-\sqrt{1})+\cdots+(\sqrt{n+1}-\sqrt{n})=\sqrt{n+1}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sqrt{n+1}=\infty$$

$\Rightarrow \sum_{n=0}^{\infty}(\sqrt{n+1}-\sqrt{n})$ diverges.

b. $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}=\frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}=\sqrt{n+1}-\sqrt{n}$$

$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by Part a.

c. $\sum_{n=0}^{\infty}(\sqrt{n+1}-\sqrt{n})^2$

$$c=\lim_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})^2}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^2}{\frac{1}{n}}=\left(\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right)^2=\left(\frac{1}{2}\right)^2=\frac{1}{4}$$

Since $c=\frac{1}{4}>0$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (the harmonic series),

$\sum_{n=0}^{\infty}(\sqrt{n+1}-\sqrt{n})^2$ diverges by LCT.

3. Determine whether each of the following series converges or diverges.

In this question you may use the fact that $\lim_{x\to 0}(\cos x)^{1/x^2} = e^{-1/2}$.

a. $\sum_{n=1}^{\infty} (\cos(1/n))^{n^2}$

$$a_n = (\cos(1/n))^{n^2} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\cos(1/n))^{n^2} = \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2} \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (\cos(1/n))^{n^2} \text{ diverges by nTT.}$$

b. $\sum_{n=1}^{\infty} (\cos(1/n))^{n^3}$

$$a_n = (\cos(1/n))^{n^3} \Rightarrow L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\cos(1/n)^{n^3}\right|^{1/n} = \lim_{n\to\infty} (\cos(1/n))^{n^2}$$

$$= \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

$$L = e^{-1/2} < 1 \Rightarrow \sum_{n=1}^{\infty} (\cos(1/n))^{n^3} \text{ converges by nRT.}$$

c. $\sum_{n=1}^{\infty} \ln(\cos(1/n)) = -\sum_{n=1}^{\infty} \left(-\ln(\cos(\tfrac{1}{n}))\right)$

$$c = \lim_{n\to\infty} \frac{-\ln(\cos(1/n))}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(-n^2 \ln(\cos(1/n))\right) = \lim_{n\to\infty} \left(-\ln\left((\cos(1/n))^{n^2}\right)\right)$$

$$= \lim_{x\to 0} \left(-\ln\left((\cos(x))^{1/x^2}\right)\right) = -\ln(e^{-1/2}) = \frac{1}{2}$$

Since $c = \frac{1}{2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$),

$\sum_{n=1}^{\infty} \left(-\ln(\cos(\tfrac{1}{n}))\right)$ converges by LCT.

Hence $\sum_{n=1}^{\infty} \ln(\cos(\tfrac{1}{n}))$ converges.

4a. A sequence $\{a_n\}_{n=0}^{\infty}$ satisfies $a_0 = 2$ and $a_n = 3 - \dfrac{2}{(a_{n-1})^2}$ for $n \geq 1$.

i. Fill in the boxes: $a_1 = \boxed{\frac{5}{2}}$ and $a_2 = \boxed{\frac{67}{25}}$

ii. Show that the sequence $\{a_n\}_{n=0}^{\infty}$ converges and find its limit.

$\frac{2}{a_{n-1}^2} > 0 \Rightarrow a_n = 3 - \frac{2}{a_{n-1}^2} < 3$ for all $n \geq 1 \Rightarrow \{a_n\}_{n=0}^{\infty}$ is bounded above.

$\circledast$ $1 \leq a_0 < a_1$

$\circledast\circledast$ If $1 \leq a_k < a_{k+1}$ for some $k \geq 0$, then $1 \leq a_k^2 < a_{k+1}^2 \Rightarrow 1 \geq \frac{1}{a_k^2} > \frac{1}{a_{k+1}^2}$

$\Rightarrow -2 \leq -\frac{2}{a_k^2} < -\frac{2}{a_{k+1}^2} \Rightarrow 1 \leq 3 - \frac{2}{a_k^2} < 3 - \frac{2}{a_{k+1}^2} \Rightarrow 1 \leq a_{k+1} < a_{k+2}$

$\Rightarrow \{a_n\}_{n=0}^{\infty}$ is increasing.

$\{a_n\}_{n=0}^{\infty}$ is bounded above and increasing $\Rightarrow \{a_n\}_{n=0}^{\infty}$ converges by MST.

Let $L = \lim_{n\to\infty} a_n$. $a_n = 3 - \frac{2}{a_{n-1}^2}$ for $n \geq 1 \Rightarrow L = 3 - \frac{2}{L^2} \Rightarrow L^3 - 3L^2 + 2 = 0$

$\Rightarrow (L-1)(L^2 - 2L - 2) = 0 \Rightarrow L = 1,\ L = 1 - \sqrt{3}$ or $L = 1 + \sqrt{3}$.

$a_n \geq 2$ for $n \geq 0 \Rightarrow L \geq 2 \Rightarrow L = 1 + \sqrt{3}$

4b. A sequence $\{c_n\}_{n=0}^{\infty}$ satisfies $c_0 = 1$, $c_1 = 2$, and $c_{n+1} = 3c_n - \dfrac{2(c_{n-1})^2}{c_n}$ for $n \geq 1$.

i. Fill in the boxes: $c_2 = \boxed{5}$ and $c_3 = \boxed{\frac{67}{5}}$

ii. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$.

Let $a_n = \frac{c_{n+1}}{c_n}$ for $n \geq 0$. Then $\{a_n\}_{n=0}^{\infty}$ is the sequence in Part a.

$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} |a_n| = 1 + \sqrt{3}$ by Part a.

Hence $R = \frac{1}{1+\sqrt{3}}$.