1. Consider the parametric curve

$$\mathscr{C}_q: \ \mathbf{r}(t) = (qt^3 + t + 4)\,\mathbf{i} + (qt^2 + t - 3)\,\mathbf{j} + (t^3 + t^2 + t)\,\mathbf{k}, \quad (-\infty < t < \infty),$$

where q is a constant.

a. Find all points where the curve $\mathscr{C}_5$ intersects the plane $x + 4y - 5z = 7$. [Note that in this part $q = 5$.]

$$\left.\begin{aligned} &\vec{r}(t) = (5t^3+t+4)\vec{i} + (5t^2+t-3)\vec{j} + (t^3+t^2+t)\vec{k} \\ &x+4y-5z=7 \end{aligned}\right\}$$

$\Rightarrow (5t^3+t+4) + 4\cdot(5t^2+t-3) - 5\cdot(t^3+t^2+t) = 7$

$\Rightarrow 15t^2 = 15 \Rightarrow t^2 = 1 \Rightarrow t=1$ or $t=-1$

$\Downarrow \qquad \Downarrow$

$(x,y,z) = (10, 3, 3)\ ,\ (-2, 1, -1)$

b. Determine all values of the constant q for which the curve $\mathscr{C}_q$ lies in a plane, and for each of these values, find an equation for the corresponding plane.

$\mathscr{C}_q$ lies in the plane $ax+by+cz=d$ where a,b,c (not all 0) and d are constants

$\Downarrow$

$a\cdot(qt^3+t+4) + b\cdot(qt^2+t-3) + c\cdot(t^3+t^2+t) = d$ for all t

$\Downarrow$

$(aq+c)t^3 + (bq+c)t^2 + (a+b+c)t + (4a-3b-d) = 0$ for all t

$\Downarrow$

$$\left.\begin{aligned} &(1)\ aq+c=0 \\ &(2)\ bq+c=0 \\ &(3)\ a+b+c=0 \\ &(4)\ 4a-3b-d=0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &(1)-(2) \Rightarrow (a-b)\cdot q = 0 \Rightarrow (b=a \text{ or } q=0) \\ &\qquad\qquad\qquad \text{and} \\ &(1)+(2)-2\times(3) \Rightarrow (a+b)\cdot(q-2)=0 \Rightarrow (b=-a \text{ or } q=2) \end{aligned}\right\}$$

If $b=a$ and $b=-a$, then $a=b=0 \overset{(3)}{\Rightarrow} c=0$, contradiction.

If $q=0$, then $b=-a \Rightarrow c=0$, $d=7a$.

The plane $x-y=7$ contains $\mathscr{C}_0$.

If $q=2$, then $b=a \Rightarrow c=-2a$, $d=a$.

The plane $x+y-2z=1$ contains $\mathscr{C}_2$.

2. Evaluate the following limits.

a. $\lim_{(x,y)\to(0,0)} \frac{x^2y^7}{x^6+y^{10}} = 0$

$\frac{a}{c} + \frac{b}{d} = \frac{2}{6} + \frac{7}{10} = \frac{31}{30} > 1 \Rightarrow$ The limit is 0 by Sertöz Theorem

b. $\lim_{(x,y)\to(0,0)} \frac{xy(x-y)}{x^2+y^2} = \underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}}_{①} - \underbrace{\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}}_{②} = 0 - 0 = 0$

$\frac{a}{c} + \frac{b}{d} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} > 1 \Rightarrow$ ① is 0 by Sertöz Theorem.

$\frac{a}{c} + \frac{b}{d} = \frac{1}{2} + \frac{2}{2} = \frac{3}{2} > 1 \Rightarrow$ ② is 0 by Sertöz Theorem.

c. $\lim_{(x,y)\to(0,0)} \frac{xy(x-y)^2}{x^4+y^4}$

$\lim_{(x,y)\to(0,0)} \frac{xy\cdot(x-y)^2}{x^4+y^4} = \lim_{x\to 0} \frac{x\cdot 0\cdot(x-0)^2}{x^4+0^4} = \lim_{x\to 0} \frac{0}{x^4} = \lim_{x\to 0} 0 = 0$ #
along the x-axis

$\lim_{(x,y)\to(0,0)} \frac{xy\cdot(x-y)^2}{x^4+y^4} = \lim_{x\to 0} \frac{x\cdot(-x)\cdot(x-(-x))^2}{x^4+(-x)^4} = \lim_{x\to 0} \frac{-4x^4}{2x^4} = \lim_{x\to 0} (-2) = -2$
along the line $y=-x$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy(x-y)^2}{x^4+y^4}$ does not exist by 2-Path Test.

3. Suppose $f(x, y, z)$ is a differentiable function with $\nabla f(P_0) = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ at the point $P_0(2, 1, 1/2)$.

[Do not assume anything about the function f beyond what is given in the sentence above.]

a. Find the directional derivative of f at P_0 in the direction of the vector $\mathbf{A} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

The directional derivative is $D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = (3\vec{i} - \vec{j} - 2\vec{k}) \cdot \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

where $\vec{u} = (\text{the direction of } \vec{A}) = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}$

b. Find $D_{\mathbf{u}} f(P_0)$ if $\mathbf{u}$ is a unit vector which makes an angle of $120°$ with $\nabla f(P_0)$.

$$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = \underbrace{|\vec{\nabla} f(P_0)|}_{=\sqrt{3^2+(-1)^2+(-2)^2} = \sqrt{14}} \underbrace{|\vec{u}|}_{=1} \underbrace{\cos\theta}_{=\cos 120° = -\frac{1}{2}} = \sqrt{14} \cdot 1 \cdot \left(-\frac{1}{2}\right) = -\sqrt{\frac{7}{2}}$$

c. Find an equation for the tangent plane to the level surface of f passing through the point P_0.

$$3 \cdot (x-2) + (-1) \cdot (y-1) + (-2) \cdot \left(z - \frac{1}{2}\right) = 0$$

OR

$$3x - y - 2z = 4$$

d. Find $\nabla g(2, 1/2)$ if $g(x, y) = f(x^2 y, xy, xy^2)$.

$$g_x = f_1 \cdot 2xy + f_2 \cdot y + f_3 \cdot y^2$$

$$\Rightarrow g_x\left(2, \tfrac{1}{2}\right) = f_1(2, 1, 1/2) \cdot 2 + f_2(2, 1, 1/2) \cdot \tfrac{1}{2} + f_3(2, 1, 1/2) \cdot \tfrac{1}{4}$$

$$= 3 \cdot 2 + (-1) \cdot \tfrac{1}{2} + (-2) \cdot \tfrac{1}{4} = 5$$

$$g_y = f_1 \cdot x^2 + f_2 \cdot x + f_3 \cdot 2xy$$

$$\Rightarrow g_y\left(2, \tfrac{1}{2}\right) = f_1(2, 1, 1/2) \cdot 4 + f_2(2, 1, 1/2) \cdot 2 + f_3(2, 1, 1/2) \cdot 2$$

$$= 3 \cdot 4 + (-1) \cdot 2 + (-2) \cdot 2 = 6$$

Hence, $\vec{\nabla} g\left(2, \tfrac{1}{2}\right) = 5\vec{i} + 6\vec{j}$

4. Find and classify the critical points of the function $f(x,y) = 4x + 2xy - 2xy^2 - 3x^2$.

$$\left.\begin{array}{l} ①\ f_x = 4 + 2y - 2y^2 - 6x = 0 \\ ②\ f_y = 2x - 4xy = 0 \end{array}\right\} \xrightarrow{②} 2x(1-2y) = 0 \Rightarrow x = 0 \text{ or } y = \frac{1}{2}$$

$$① \text{ and } x = 0 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = 2 \text{ or } y = -1$$

$$\Downarrow$$

$$(x,y) = (0,2),\ (0,-1)$$

$$① \text{ and } y = \frac{1}{2} \Rightarrow \frac{9}{2} - 6x = 0 \Rightarrow x = \frac{3}{4}$$

$$\Downarrow$$

$$(x,y) = \left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6 & 2-4y \\ 2-4y & -4x \end{vmatrix}$$

$$\Delta(0,2) = \begin{vmatrix} -6 & -6 \\ -6 & 0 \end{vmatrix} = (-6)\cdot 0 - (-6)^2 = -36 < 0 \Rightarrow (0,2) \text{ is a saddle point}$$

$$\Delta(0,-1) = \begin{vmatrix} -6 & 6 \\ 6 & 0 \end{vmatrix} = (-6)\cdot 0 - 6^2 = -36 < 0 \Rightarrow (0,-1) \text{ is a saddle point}$$

$$\left.\begin{array}{l} \Delta\left(\frac{3}{4}, \frac{1}{2}\right) = \begin{vmatrix} -6 & 0 \\ 0 & -3 \end{vmatrix} = (-6)\cdot(-3) - 0^2 = 18 > 0 \\ \text{and } f_{xx}\left(\frac{3}{4}, \frac{1}{2}\right) = -6 < 0 \end{array}\right\} \Rightarrow \left(\frac{3}{4}, \frac{1}{2}\right) \text{ is a local max}$$

$$\left.\begin{array}{rl} \Delta > 0 \text{ and } f_{xx} > 0 & \Longrightarrow \text{local minimum} \\ \Delta > 0 \text{ and } f_{xx} < 0 & \Longrightarrow \text{local maximum} \\ \Delta < 0 & \Longrightarrow \text{saddle point} \end{array}\right\} \text{ where } \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$