Bilkent University

Quiz # 01
Math 102 Section 09 Calculus II
12 February 2024 Monday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1)

(a) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$ converge or diverge? If it converges find its sum.

(b) Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge or diverge? If it converges find its sum.

(c) Does the sequence $\displaystyle\left(\frac{n}{n+1}\right)^n$ converge or diverge? If it converges find its limit.

Show your work in detail. Correct answers with no justification will not get any credit.

Grading: 2+5+3=10 points

Solution: (Grader: `melis.gezer@bilkent.edu.tr`)

(a) From the list of useful limits we know that $\displaystyle\lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$. Since the general term of this series does not converge to zero the series diverges by the n-th Term Test.

(b) By the partial fractions technique we find that

$$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right].$$

Then the sequence of partial sums of this series is of the form

$$s_n = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)\left(+\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right]$$

and after simplifying we get

$$s_n = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right]. \quad \text{Thus } \lim_{n\to\infty} s_n = \frac{3}{4}.$$

(c) Note that $a_n = \displaystyle\left(\frac{n}{n+1}\right)^n = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e}$ as $n \to \infty$, which we again recall from the list of useful limits.