Bilkent University

Quiz # 02
Math 102 Section 09 Calculus II
19 February 2024 Monday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) (a) Assume the fact that the function $f(x) = \frac{\ln x}{x^2}$ is positive, continuous and decreasing for $x \geq 3$. Use the Integral Test to decide if the series $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

(b) Use the Direct Comparison Test to decide if the series $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

(c) Use the Limit Comparison Test to decide if the series $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

Show your work in detail. Correct answers with no justification will not get any credit.

Grading: 4+3+3=10 points

Solution: (Grader: `melis.gezer@bilkent.edu.tr`)

(a)

$$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\left(\frac{1+\ln x}{x}\right),$$

where we used integration by parts with $u = \ln x$ and $dv = dx/x^2$. Then

$$\int_3^{\infty} \frac{\ln x}{x^2}\, dx = -\lim_{R\to\infty} \left(\frac{1+\ln x}{x} \Bigg|_3^R \right) = \frac{1+\ln 3}{3} < \infty.$$

Hence by the Integral Test $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges.

(b) Since $\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = 0$ for any $\alpha > 0$, we have $\ln x < x^{\alpha}$ for all large x. Let $\alpha = 1/2$. Since

$$0 < \frac{\ln n}{n^2} < \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}, \quad \text{for all large } n,$$

and since $\sum \frac{1}{n^{3/2}}$ converges by p-test, our series $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges by the Direct Comparison Test.

(c) Let $a_n = \frac{\ln n}{n^2}$ and $b_n = \frac{1}{n^{3/2}}$. Then since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = 0,$$

and since $\sum b_n$ converges by p-test, our series $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges by the Limit Comparison Test.