Bilkent University

Quiz # 05
Math 102 Section 09 Calculus II
11 March 2024 Monday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Assume that y is an analytic function of x,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots ,$$

and satisfies the initial value problem

$$y'' + 4y = 0,\ y(0) = 1,\ y'(0) = 0.$$

Show by induction that

$$a_{2n+1} = 0 \text{ and } a_{2n} = (-1)^n \frac{4^n}{(2n)!}, \text{ where } n = 0, 1, 2, \ldots .$$

Then evaluate $y(\pi/6)$.

Show your work in detail. Correct answers with no justification will not get any credit.

Grading: 8+2=10 points

Solution: (Grader: `melis.gezer@bilkent.edu.tr`)

We have

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_{n+2}x^{n+2} + \cdots , \\ y' &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + (n+2)a_{n+2}x^{n+1} + \ldots , \\ y'' &= 1\cdot 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots . \end{aligned}$$

Thus

$$y''+4y = (1\cdot 2a_2+4a_0)+(2\cdot 3a_3+4a_1)x+(3\cdot 4a_4+4a_2)x^2+\cdots+((n+1)(n+2)a_{n+2}+4a_n)x^n+\cdots = 0,$$

and hence

$$a_{n+2} = -\frac{4}{(n+1)(n+2)}\, a_n. \qquad (*)$$

Also from the initial conditions we see that

$$a_0 = 1 \text{ and } a_1 = 0.$$

To prove that odd indexed coefficients are zero first observe that $a_1 = 0$. For induction hypothesis assume that $a_{2n+1} = 0$ for some $n \geq 0$. Now check that using the equation $(*)$,

$$a_{2n+3} = -\frac{4}{(2n+2)(2n+3)}a_{2n+1} = 0, \text{ since } a_{2n+1} = 0 \text{ by assumption.}$$

This proves that all odd indexed coefficients are zero.

For the even indexed coefficients first observe that $a_0 = 1$ is of the claimed form. Now assume the claim holds for some $n \geq 0$, and again using the equation $(*)$ check that

$$a_{2n+2} = -\frac{4}{(2n+1)(2n+2)} a_{2n} = -\frac{4}{(2n+1)(2n+2)} (-1)^n \frac{4^n}{(2n)!} = (-1)^{n+1} \frac{4^{n+1}}{(2n+2)!},$$

which is of the required form. This completes the induction. Hence we have

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n}.$$

We recognize this function as

$$y(x) = \cos 2x,$$

and hence

$$y(\pi/6) = \cos(\pi/3) = \frac{1}{2}.$$