Math 112 Intermediate Calculus II – Midterm Exam I-Solutions

Date: 18 June 2004, Friday — Instructor: Ali Sinan Sertöz

Q-1) i) $f(x) = (\tan x)^{(\sec x)}$. Find $f'(x)$ and $f'(\pi/4)$.

ii) $f(x) = x^{2x+1} + \ln\ln(x^3+1)$. Find $f'(x)$ and $f'(1)$.

i) $f'(x) = (\tan x)^{(\sec x)} \left(\sec x \ \tan x \ \ln(\tan x) + \sec x \, \dfrac{\sec^2 x}{\tan x} \right)$, $f'(\frac{\pi}{4}) = 2\sqrt{2}$.

ii) $f'(x) = x^{2x+1} \left(2 \ln x + \dfrac{2x+1}{x} \right) + \dfrac{1}{\ln(x^3+1)} \, \dfrac{1}{x^3+1} \, (3x^2)$, $f'(1) = 3 + \dfrac{3}{\ln 4}$.

Q-2) i) Evaluate $\displaystyle\int x^2 \arctan x \, dx$.

ii) Evaluate $\displaystyle\int x^5 \, e^x \, dx$.

i) Use by-parts with $u = \arctan x$ to get

$$\begin{aligned}
\int x^2 \arctan x \, dx &= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2} \right) \, dx \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) \right) + C \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C
\end{aligned}$$

ii) Use tabular integration (see page 552) to get

$$\int x^5 \, e^x \, dx = e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C.$$

Q-3) Evaluate $\displaystyle\int \frac{2x^3 + 8x^2 + 11x + 6}{(x+1)^2(x^2+2x+2)}\, dx$.

$$\frac{2x^3 + 8x^2 + 11x + 6}{(x+1)^2(x^2+2x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+2}.$$

By multiplying both sides with $(x+1)^2$ and evaluating at $x = -1$ we immediately find that $B = 1$. Bringing to common denominator and equating numerators of both sides we find that A, C, D must satisfy

$A + C = 2,$
$3A + 2C + D + 1 = 8,$
$4A + C + 2D + 2 = 11,$
$2A + D + 2 = 6.$

From the first and last equations solve for C and D, respectively, in terms of A. Substitute into the second equation to find $A = 1$. This then immediately gives $C = 1$ and $D = 2$.

$$\begin{aligned}\frac{2x^3+8x^2+11x+6}{(x+1)^2(x^2+2x+2)} &= \frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{x+2}{x^2+2x+2} \\ &= \frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{2}\frac{2x+2}{x^2+2x+2}+\frac{1}{(x+1)^2+1}\end{aligned}$$

Integrating both sides we get

$$\int \frac{2x^3+8x^2+11x+6}{(x+1)^2(x^2+2x+2)}\,dx = \ln|x+1| - \frac{1}{x+1} + \frac{1}{2}\ln(x^2+2x+2) + \arctan(x+1) + C.$$

Q-4) Evaluate $\displaystyle\int \frac{dx}{1+\sqrt{1-x^2}}$.

First put $x = \sin\theta$ and then put $z = \tan\frac{\theta}{2}$ (see page 570) to get

$$\int \frac{dx}{1+\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{1+\cos\theta} = \int \frac{1-z^2}{1+z^2}\,dz = \int \left(\frac{2}{1+z^2}-1\right)\,dz$$

$$= 2\arctan z - z + C = \theta - \tan\frac{\theta}{2} + C = \arcsin x - \frac{x}{1+\sqrt{1-x^2}} + C.$$

Q-5) Check for convergence. If possible evaluate the exact value.

i) $\displaystyle\int_1^\infty \frac{37x^9+16x^6+x^3}{2e^{x^2}+e^{2x}+3\sin x}\,dx.$

ii) $\displaystyle\int_1^\infty \frac{3}{2x^2+5x+2}\,dx.$

i) Let $f(x) = \dfrac{37x^9+16x^6+x^3}{2e^{x^2}+e^{2x}+3\sin x}$ and $g(x) = e^{-x}$.

$\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$. So for large x we have $0 \le f(x) < g(x)$. Since $\int_1^\infty g(x)dx$ converges, the given integral also converges by the comparison test.

ii) We immediately see that this integral converges by limit comparing with $1/x^2$. Moreover we can evaluate the integral:

$$\begin{aligned}\int_1^\infty \frac{3}{2x^2+5x+2}\,dx &= \int_1^\infty \left(\frac{2}{2x+1}-\frac{1}{x+2}\right)\,dx \\ &= \left(\ln|2x+1| - \ln|x+2|\Big|_1^\infty\right) \\ &= \left(\ln\frac{|2x+1|}{|x+2|}\Bigg|_1^\infty\right) \\ &= \ln 2.\end{aligned}$$