MATH 113 Solutions manual for Homework IV

1. Page 187 Exercise 9:
A function f, continuous on $[a, b]$, has a second derivative f'' everywhere on the open interval (a, b). The line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of f at a third point $(c, f(c))$, where $a < c < b$. Prove that $f''(t) = 0$ for at least one point t in (a, b).

Solution:
Let L be the line through the points $(a, f(a))$ and $(b, f(b))$. Then the equation of L can be written as

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \quad \text{(I)}$$

and

$$y - f(b) = \frac{f(b) - f(a)}{b - a}(x - b) \quad \text{(II)}$$

$(c, f(c))$ is on this line. So by (I),

$$f(c) - f(a) = \frac{f(b) - f(a)}{b - a}(c - a) \Rightarrow \frac{f(c) - f(a)}{c - a} = \frac{f(b) - f(a)}{b - a}. \quad (*)$$

By (II)

$$f(c) - f(b) = \frac{f(b) - f(a)}{b - a}(c - b) \Rightarrow \frac{f(c) - f(b)}{c - b} = \frac{f(b) - f(a)}{b - a}. \quad (**)$$

First we apply the Mean Value Theorem to f on $[a, c]$. There is $c_1 \in (a, c)$ such that

$$f'(c_1) = \frac{f(c) - f(a)}{c - a} \overset{(*)}{=} \frac{f(b) - f(a)}{b - a}.$$

Next we apply the Mean Value Theorem to f on $[c, b]$. There is $c_2 \in (c, b)$ such that

$$f'(c_2) = \frac{f(b) - f(c)}{b - c} \overset{(**)}{=} \frac{f(b) - f(a)}{b - a}.$$

So $f'(c_1) = f'(c_2)$. Then we apply Rolle's Theorem to f' on $[c_1, c_2]$. Note that f' is differentiable on (c_1, c_2) (derivative of f' is f'') and f' is continuous on $[c_1, c_2]$ (f' is differentiable on (a, b) implies f' is continuous on (a, b).) So by Rolle's Theorem there is $t \in (c_1, c_2) \subset (a, b)$ such that $(f')'(t) = 0$, i.e. $f''(t) = 0$.

2. Page 191 Exercise 9.
Sketch the graph of $f(x) = \dfrac{x}{1 + x^2}$.

Solution:
There is no vertical asymptote. $\lim_{x \to \pm\infty} \frac{x}{1+x^2} = 0$. So the line $y = 0$ is horizontal asymptote as $x \to \pm\infty$.

$$f'(x) = \frac{1 + x^2 - x \cdot 2x}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2} = 0 \Rightarrow x = \pm 1.$$

x		-1		1	
$1-x^2$	$-$	0	$+$	0	$-$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘		↗		↘

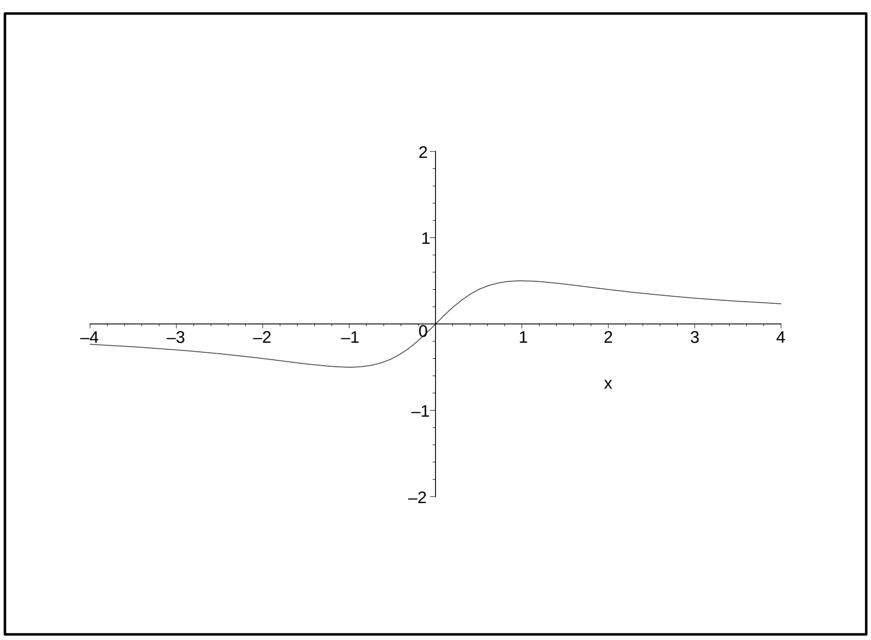
$$f''(x) = \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)2x}{(1+x^2)^4} = \frac{2x^3-6x}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$2x$	$-$		$-$	0	$+$		$+$
x^2-3	$+$	0	$-$		$-$	0	$+$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\cap$		$\cup$		$\cap$		$\cup$

The complete table is

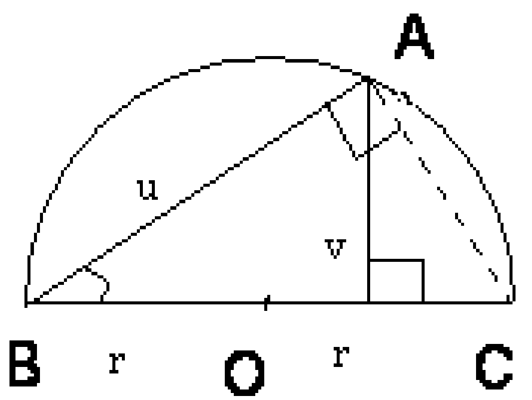
x	$-\infty$		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		$+\infty$
$f'(x)$		$-$		$-$	0	$+$		$+$	0	$-$		$-$	
$f''(x)$		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
$f(x)$	0		$-\frac{\sqrt{3}}{4}$		$-\frac{1}{2}$		0		$\frac{1}{2}$		$\frac{\sqrt{3}}{4}$		0

The graph is then given below:



3-a. Page 195 Exercise 22. a)

An isosceles triangle is inscribed in a circle of radius r as shown in the figure below. If the angle 2α at the apex is restricted to lie betwen 0 and $\pi/2$, find the largest value and the smallest value of the perimeter of the triangle. Give full details of your reasoning.



Solution:

Let P be the perimeter of the triangle. Then $P = 2(u + v)$. $u/2r = \cos\alpha$, so $u = 2r\cos\alpha$ and $v/u = \sin\alpha$, so $v = u\sin\alpha = 2r\cos\alpha\sin\alpha$. Thus $P = 4r(\cos\alpha + \cos\alpha\sin\alpha)$ and we want to find the maximum and minimum values of P when $0 \le \alpha \le \frac{\pi}{4}$ (given interval).

$$\frac{dP}{d\alpha} = 4r(-\sin\alpha \underbrace{-\sin^2\alpha + \cos^2\alpha}_{\cos 2\alpha = 1 - 2\sin^2\alpha}) = -4r(2\sin^2\alpha + \sin\alpha - 1) = 0 \Rightarrow \sin\alpha = \frac{1}{2}, \sin\alpha = -1.$$

In the given domain $\sin\alpha \ge 0$, so $\sin\alpha = \frac{1}{2}$, i.e. $\alpha = \frac{\pi}{6}$ is the only critical point. We check $\alpha = \frac{\pi}{6}$, $\alpha = 0$ and $\alpha = \frac{\pi}{4}$.

$$\begin{aligned}
\alpha = \frac{\pi}{6} \quad &\Rightarrow \quad P = 4r\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) = 3\sqrt{3}r \quad \text{(largest)} \\
\alpha = 0 \quad &\Rightarrow \quad P = 4r \quad \text{(smallest)} \\
\alpha = \frac{\pi}{4} \quad &\Rightarrow \quad P = 4r\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}r
\end{aligned}$$

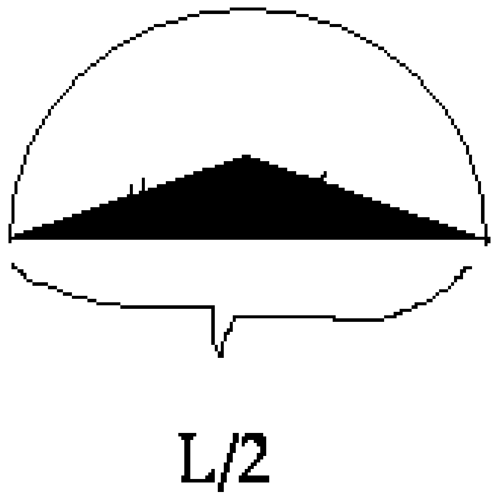
So P has maximum at $\alpha = \frac{\pi}{6}$ and $P_{\max} = 3\sqrt{3}r$, P has minimum at $\alpha = 0$ and $P_{\min} = 4r$.

3-b. Page 196 Exercise 22. b)

What is the radius of the smallest circular disk large enough to cover every isosceles triangle of a given perimeter L?

Solution:

Let $f(\alpha) = \cos\alpha + \cos\alpha\sin\alpha$. Then by a) $P = 4rf(\alpha)$. In a) r was constant and we actually found max. and min. of $f(\alpha)$ when $0 \le \alpha \le \frac{\pi}{4}$. In this part $P = L$ is constant, so $r = \frac{L}{4}f(\alpha)$. When $0 \le \alpha \le \frac{\pi}{4}$, $f(\alpha)$ has minimum at $\alpha = 0$, so r has maximum at $\alpha = 0$. So the largest radius for the circumscribing circle for $0 \le \alpha \le \frac{\pi}{4}$ is $r = \frac{L}{4f(0)} = \frac{L}{4}$, i.e. the radius of the smallest circular disk which covers all isosceles triangles with all three vertices on the circle and $0 \le \alpha \le \frac{\pi}{4}$ and perimeter=L is $\frac{L}{4}$. What about isosceles triangles with perimeter=L and $0 \le \alpha \le \frac{\pi}{4}$? In this case we put them into the disk as in the figure



Since the base length$\le \frac{L}{2}$, this is possible. Thus the smallest radius in all cases is $r = \frac{L}{4}$.