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Instructor: Ali Sinan Sertöz

Exercise 4, page 60 of Apostol's Calculus:

A point (x, y) in the plane is called a *lattice* point if both coordinates x and y are integers. Let P be a polygon whose vertices are lattice points. The area of P is $I + \frac{1}{2}B - 1$, where I denotes the number of lattice points inside the polygon and B denotes the number on the boundary.
(a) Prove that the formula is valid for rectangles with sides parallel to the coordinate axes.
(b) Prove that the formula is valid for right triangles and parallelograms.
(c) Use induction on the number of edges to construct a proof for general polygons.

Solution:

(a) Let $ABCD$ be a rectangle with sides parallel to the coordinate axes. Without loss of generality we may assume that $A = (0, 0)$. Then let $B = (m, 0)$, $C = (m, n)$ and $D = (0, n)$, where $m, n \in \mathbb{N}$. The area of this rectangle is clearly mn. We can easily check that $I = (m - 1)(n - 1)$, $B = 2(m + n)$. Then $I + \frac{1}{2}B - 1$ gives the area.

(b) Now let us calculate the area of the right triangle ABC where the points are as given in the first part. The area of this triangle is clearly $\frac{1}{2}mn$. Let x be the number of lattice points on the hypothenuse AC, minus 2, i.e. x counts the lattice points on AC lying strictly between A and C. To calculate the interior lattice points of this right triangle we first recall that there are $(m - 1)(n - 1)$ interior points in the rectangle $ABCD$, and x of these lie on the hypothenuse. Half of the remaining interior points lie in the triangle ABC and the other half lies in the right triangle ACD. So for this triangle $I = \frac{1}{2}((m - 1)(n - 1) - x)$, $B = m + n + x + 1$. Then $I + \frac{1}{2}B - 1$ gives the area.

Given an arbitrary triangle, first complete it to a rectangle whose sides are parallel to the axes by adjoining some right triangles whose legs are parallel to the axes.

For example let ABC be a triangle with $A = (0, 0)$, $B = (m, n)$, $C = (r, s)$. Assume $r < m$ and $s > n$. The other cases are treated similarly.

Let us give names to the vertices of the rectangle obtained by adjoining to ABC some right triangles whose legs are parallel to the axes; $D = (m, 0)$, $E = (m, s)$, $F = (0, s)$. We thus obtain the rectangle $ADEF$.

Let x, y, z denote the number of lattice points on the lines AB, BC and CA respectively, not counting the points A, B, C in each case.

For the triangle ADB, the number of lattice points on the interior is $I_1 = \frac{1}{2}((m-1)(n-1)-x)$, the number of lattice points on the boundary is $B_1 = m + n + x + 1$.

For the triangle BEC, the number of lattice points on the interior is $I_2 = \frac{1}{2}((s - n - 1)(m - r - 1) - y)$, the number of lattice points on the boundary is $B_2 = (s - n) + (m - r) + y + 1$.

For the triangle ACF, the number of lattice points on the interior is $I_3 = \frac{1}{2}((r - 1)(s - 1) - z)$, the number of lattice points on the boundary is $B_3 = r + s + z + 1$.

From the area of the triangle ABC, we subtract the areas of these three right triangles from the area of the big rectangle.

Area of ABC=$ms - \left[\left(I_1 + \frac{1}{2}B_1 - 1\right) + \left(I_2 + \frac{1}{2}B_2 - 1\right) + \left(I_3 + \frac{1}{2}B_3 - 1\right)\right] = \frac{ms - nr - 2}{2}$.

Note that in this calculation we could also write the areas of the right triangles directly as 1/2 times base times height.

Now back to triangle ABC. The number of its interior lattice points is $I = (m-1)(n-1) - (I_1 + I_2 + I_3 + x + y + z)$. The boundary lattice points are $B = x + y + z + 3$. We calculate that $I + \frac{1}{2}B - 1 = \frac{ms - nr - 2}{2}$, proving the area formula for any triangle.

(c) Let $A_1 \cdots A_{n+1}$ be a polygon of $n+1$ sides where each of the vertices $A_1, \ldots, A_{n+1}$ is a lattice point.

Let x denote the number of lattice points on the line A_1A_{n+1}, not counting the end points.

Let Δ_{n+1} denote the area of the polygon $A_1 \cdots A_{n+1}$. For this polygon let I_{n+1} denote the number of interior lattice points and B_{n+1} denote the number of boundary lattice points.

Let Δ_n denote the area of the polygon $A_1 \cdots A_n$. For this polygon let I_n denote the number of interior lattice points and B_n denote the number of boundary lattice points.

Let Δ_3 denote the area of the triangle $A_1A_nA_{n+1}$. For this triangle let I_3 denote the number of interior lattice points and B_3 denote the number of boundary lattice points.

We already proved that $\Delta_3 = I_3 + \frac{1}{2}B_3 - 1$.

Assume that $\Delta_n = I_n + \frac{1}{2}B_n - 1$. We want to derive the formula for Δ_{n+1}.

Observe that $I_{n+1} = I_3 + I_n + x$ and $B_{n+1} = (B_3 - x) + (B_n - x) - 2 = B_3 + B_n - 2x - 2$.

Finally we have

$$\begin{aligned}
\Delta_{n+1} &= \Delta_3 + \Delta_n \\
&= (I_3 + \frac{1}{2}B_3 - 1) + (I_n + \frac{1}{2}B_n - 1) \\
&= (I_3 + \frac{1}{2}B_3 - 1) + (I_n + \frac{1}{2}B_n - 1) + [x + 1 + \frac{1}{2}(-2x - 2)] \\
&= (I_3 + I_n + x) + \frac{1}{2}(B_3 + B_n - 2x - 2) - 1 \\
&= I_{n+1} + \frac{1}{2}B_{n+1} - 1,
\end{aligned}$$

completing the induction argument and proving the formula.

`send comments to sertoz@bilkent.edu.tr`