Date: 5 January 2005, Wednesday
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Math 113 Calculus – Final Exam – Solutions

Q-1) Write the derivatives of the following functions with respect to x. Do not simplify your answers. No partial credits!

a) $f(x) = \ln\cos x + 3^{\cos x}$, $f'(x) = \dfrac{1}{\cos x}(-\sin x) + 3^{\cos x}\,(-\sin x\ \ln 3)\,.$

b) $f(x) = x^{1/x}$, $f'(x) = x^{1/x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\dfrac{1}{x}\right).$

c) $f(x) = x^{3^x}$, $f'(x) = x^{3^x}\left(3^x\dfrac{1}{x} + \ln x\ [3^x \ln 3]\right).$

d) $f(x) = \ln\sin x + x^{\cos x} + \pi^x + 2^\pi$, $f'(x) = \dfrac{1}{\sin x}\cos x + x^{\cos x}\left(-\sin x\ln x + \cos x\ \dfrac{1}{x}\right) + \pi^x\ln\pi.$

Q-2) Calculate the following derivatives. Write the answers inside the given boxes. No partial credits!

a) $f(x) = x\cos x, \quad x(t) = \dfrac{1+t}{1-t}. \quad \left.\dfrac{df}{dt}\right|_{t=-1} = ?$

$f'(x) = \cos x - x\sin x. \quad x(-1) = 0. \quad \mathbf{f'(0) = 1}. \quad x'(t) = \dfrac{2}{(1-t)^2}. \quad \mathbf{x'(-1) = \dfrac{1}{2}}.$

$$\left.\frac{df}{dt}\right|_{t=-1} = \left.\frac{df}{dx}\right|_{x=0}\left.\frac{dx}{dt}\right|_{t=-1} = \mathbf{1}\cdot\frac{\mathbf{1}}{\mathbf{2}} = \boxed{\frac{\mathbf{1}}{\mathbf{2}}}$$

b) $f(x) = \dfrac{x^3+x}{x^7+1}, \quad x(t) = \tan t. \quad \left.\dfrac{df}{dt}\right|_{t=\pi/4} = ?$

$$f'(x) = \frac{(3x^2+1)(x^7+1)-(x^3+x)(7x^6)}{(x^7+1)^2}, \quad x(\pi/4) = 1, \quad \mathbf{f'(1) = -\frac{3}{2}}. \; x'(t) = \sec^2 t, \quad \mathbf{x'(\pi/4) = 2.}$$

$$\left.\frac{df}{dt}\right|_{t=\pi/4} = \left.\frac{df}{dx}\right|_{x=1} \left.\frac{dx}{dt}\right|_{t=\pi/4} = -\frac{\mathbf{3}}{\mathbf{2}} \cdot \mathbf{2} = \boxed{\mathbf{-3}}$$

c) $f(x) = \sin^2 x + \sinh^2 x + \ln(1+x^2), \quad x(t) = \frac{t^3-1}{t^2+1}. \quad \left.\frac{df}{dt}\right|_{t=1} = ?$

$$f'(x) = 2\sin x \cos x + 2\sinh x \cosh x + \frac{2x}{1+x^2}, \quad x(1) = 0, \quad \mathbf{f'(0) = 0.}$$

$$\left.\frac{df}{dt}\right|_{t=1} = \left.\frac{df}{dx}\right|_{x=0} \left.\frac{dx}{dt}\right|_{t=1} = \mathbf{0} \cdot \left.\frac{\mathbf{dx}}{\mathbf{dt}}\right|_{\mathbf{t=1}} = \boxed{\mathbf{0}}$$

d) $f(x) = e^x \cos x, \quad x(t) = e^t \cos t. \quad \left.\frac{df}{dt}\right|_{t=\pi/2} = ?$

$$f'(x) = e^x(\cos x - \sin x), \quad x(\pi/2) = 0, \quad \mathbf{f'(0) = 1}. \; x'(t) = e^t(\cos t - \sin t), \quad \mathbf{x'(\pi/2) = -e^{\pi/2}}.$$

$$\left.\frac{df}{dt}\right|_{t=\pi/2} = \left.\frac{df}{dx}\right|_{x=0} \left.\frac{dx}{dt}\right|_{t=\pi/2} = \mathbf{1} \cdot (\mathbf{-e^{\pi/2}}) = \boxed{\mathbf{-e^{\pi/2}}}$$

Q-3) Find the minimum and the maximum values of the function $f(x) = x^3 - 13x^2 + 48x - 41$ on $[0, 7]$.

$f'(x) = 3x^2 - 26x + 48$, $f'(x) = 0$ when $x = 6$ and $x = \frac{8}{3}$.

$f(0) = -41$, $f(8/3) = \frac{365}{27} \approx 13.51$, $f(6) = -5$, $f(7) = 1$.

Therefore the minimum value is -41 and the maximum value is $\frac{365}{27}$.

Q-4) Evaluate the integral $\int x \sin^2 x \, dx$.

Try integration by parts with $u = x$ and $dv = \sin^2 x = (1 - \cos 2x)/2$ from which $v = (x/2) - (\sin 2x)/4$. Now vdu can easily be integrated and we find $\int x \sin^2 x \; dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$.

Q-5) Find the Taylor polynomial of degree 7 of $\arctan x$ at $x = 0$.

Hint: You might want to start with $\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n + \frac{t^{n+1}}{1-t}$.

Put $t = -x^2$ and integrate from 0 to x to obtain $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$.