Math 113 Homework 3
Due: 12 November 2004 Friday class hour.
Instructor: Ali Sinan Sertöz

Q-1) Exercise 24 on page 181. A reservoir has the shape of a right circular cone. The altitude is 10 meters, and the radius of the base is 4 meters. Water is flowing into the reservoir at a constant rate of 5 cubic meters per minute. How fast is the water level rising when the depth of the water is 5 meters if (a) the vertex of the cone is up? (b) the vertex of the cone is down?

Solution:

Let r be the radius of water surface in the reservoir when the water height is h. Note that both r and h are functions of time t.

(a) We find from similar triangles that $\dfrac{r}{4} = \dfrac{10-h}{10}$, so $r = \dfrac{2}{5}(10-h)$. The volume of water in the reservoir at time t is

$$V = \frac{160\pi}{3} - \frac{\pi}{3}r^2(10-h) = \frac{160\pi}{3} - \frac{4\pi}{75}(10-h)^3.$$

Taking the derivative of V with respect to time and using the chain rule we get

$$V' = \frac{4\pi}{25}(10-h)^2 h'.$$

We know that $V' = 5$ when $h = 5$. Putting these in we find $h' = \dfrac{5}{4\pi}$.

(b) In this case $\dfrac{r}{4} = \dfrac{h}{10}$, so $r = \dfrac{2}{5}h$. The volume of water in the reservoir at time t is

$$V = \frac{\pi}{3}r^2 h = \frac{4\pi}{75}h^3.$$

Taking the derivative of V with respect to time and using the chain rule we get

$$V' = \frac{4\pi}{25}h^2 h'.$$

We know that $V' = 5$ when $h = 5$. Putting these in we find $h' = \dfrac{5}{4\pi}$.

Q-2) Exercise 1 on page 186. Show that on the graph of any quadratic polynomial the chord joining the points for which $x = a$ and $x = b$ is parallel to the tangent line at the midpoint $x = (a+b)/2$.

Solution:

Let $f(x) = c_2x^2 + c_1x + c_0$ be a general quadratic.

The slope of the chord is $m_1 = \dfrac{f(b) - f(a)}{b-a} = \dfrac{c_2(b^2-a^2) + c_1(b-a)}{b-a} = c_2(b+a) + c_1$.

The slope of the tangent line $m_2 = f'\left(\dfrac{a+b}{2}\right) = 2c_2\left(\dfrac{a+b}{2}\right) + c_1 = c_2(a+b) + c_1$.

So $m_1 = m_2$.

Q-3) Exercise 2 on page 186. Use Rolle's theorem to prove that, regardless of the value of b, there is at most one point x in the interval $-1 \leq x \leq 1$ for which $x^3 - 3x + b = 0$.

Solution:

Let $f(x) = x^3 - 3x + b$. Let $a, b \in [-1, 1]$ be two distinct points where $f(a) = f(b) = 0$. Then there is a point $c \in (a, b) \subset (-1, 1)$ where $f'(c) = 0$. But $f'(c) = 3(c^2 - 1)$ and cannot vanish at any point inside $(-1, 1)$. Therefore f can have at most one zero in this interval.

Here is an easier solution without using Rolle's theorem. Since $f'(x) = 3(x^2 - 1) < 0$ on $(-1, 1)$, then $f(x)$ is decreasing. Therefore f cannot have more than one zero. (Actually, using the intermediate value theorem for f we can conclude that f has a zero in this interval when $-2 < b < 2$.)

Q-4) Exercise 5 on page 186. Show that $x^2 = x \sin x + \cos x$ for exactly two real values of x.

Solution:

Let $f(x) = x^2 - x \sin x - \cos x$. We want to show that $f(x) = 0$ for exactly two real values of x. Since $f(-x) = f(x)$, it suffices to show that $f(x) = 0$ for exactly one value of $x > 0$.

It is easy to show that $2x \leq f'(x) \leq 3x$ for $x > 0$, so f is increasing for $x > 0$. This shows that f can have at most one zero for $x > 0$. Since $f(0) = -1$ and $f(2) \geq 1$, there exists exactly one zero for f for $x > 0$. In fact $f(\pm 1, 220468466) = 0$.

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