Date: 28 December 2005, Wednesday, Ali Sinan Sertöz

Math 113 Calculus – Final Exam – Solutions

Q-1) Do not simplify your answers in (a), (b), (c). No partial credits!

a) $f(x) = x^x$,

$f'(x) = (\exp(\ln x^x))' = (\exp(x \ln x))' = x^x(\ln x + 1)$.

b) $f(x) = \int_0^{\cosh x} \sqrt{1+t^5}\, dt$,

$f'(x) = \sqrt{1+\cosh^5 x}\ \sinh x$.

c) $f(x) = (\sec x)(\ln \tan x)$,

$$f'(x) = (\sec x \tan x)(\ln \tan x) + (\sec x)\left(\frac{\sec^2 x}{\tan x}\right).$$

d) Let $f(x) = g^{-1}(x)$ for $1 \le x \le 5$, where $g(x) = x^3 - 3x^2 + 5$. Find the slope of the tangent line to the curve $y = f(x)$ at the point $(3, 1)$.

Solution:

$f(3) = 1 \Leftrightarrow g(1) = 3$. The required slope is $f'(3)$.

$f'(3) = f'(g(1)) = 1/g'(1)$.

$g'(x) = 3x^2 - 6x$, $g'(1) = -3$, $f'(3) = -1/3$.

Q-2) Calculate the following limits:

a) $$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^3} \overset{t=1/x}{=} \lim_{t\to\pm\infty} \frac{t^3}{e^{t^2}} \overset{L'H\hat{o}pital}{=} \lim_{t\to\pm\infty} \frac{3t^2}{2te^{t^2}} = \lim_{t\to\pm\infty} \frac{3t}{2e^{t^2}} \overset{L'H\hat{o}pital}{=} \lim_{t\to\pm\infty} \frac{3}{4te^{t^2}} = 0.$$

b) $$\lim_{x\to 0} \frac{x^{21} + 21x + \sin x}{x\cos x + x} \overset{L'H\hat{o}pital}{=} \lim_{x\to 0} \frac{21x^{20} + 21 + \cos x}{\cos - x\sin x + 1} = \frac{22}{2} = 11.$$

c) $$\lim_{x\to 0} \frac{\sqrt{1+3x^2+4x^4} - \sqrt{1-5x^3-6x^5}}{x^2}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+3x^2+4x^4} - \sqrt{1-5x^3-6x^5}}{x^2} \, \frac{\sqrt{1+3x^2+4x^4} + \sqrt{1-5x^3-6x^5}}{\sqrt{1+3x^2+4x^4} + \sqrt{1-5x^3-6x^5}}$$

$$= \lim_{x\to 0} \frac{3x^2+9x^3+6x^5}{x^2\left(\sqrt{1+3x^2+4x^4}+\sqrt{1-5x^3-6x^5}\right)}$$

$$= \lim_{x\to 0} \frac{3+9x+6x^3}{\left(\sqrt{1+3x^2+4x^4}+\sqrt{1-5x^3-6x^5}\right)} = \frac{3}{2}.$$

d) $\displaystyle\lim_{x\to 0} \frac{15\tan x - 15x - 5x^3 - 2x^5}{24x\cos x - 24x + 12x^3 - x^5} \overset{Taylor}{=} \lim_{x\to 0} \frac{15(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots) - 15x - 5x^3 - 2x^5}{24x(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots) - 24x + 12x^3 - x^5}$

$$= \lim_{x\to 0} \frac{\frac{17x^7}{21}+\cdots}{-\frac{x^7}{30}+\cdots} = -\frac{170}{7}.$$

Recall: $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$

Q-3) Find the minimum and the maximum values of the function $f(x)$ on $[0,8]$, where

$$f(x) = \begin{cases} x^3-4x^2-3x+8 & \text{if } \ 0 \le x \le 5, \\ x^2-14x+63 & \text{if } \ 5 \le x \le 8. \end{cases}$$

Solution:

For $0 \le x \le 5$, $f'(x) = 3x^2 - 8x - 3 = 0$, $x = 3$ is the solution in the domain.

For $5 \le x \le 8$, $f'(x) = 2x - 14 = 0$, $x = 7$

We calculate $f(0) = 8$, $f(3) = -10$, $f(5) = 18$, $f(7) = 14$, $f(8) = 15$.

Therefore minimum of f is -10, and the maximum is 18.

Q-4) Assume that $f'(x) = \dfrac{x^2+2}{((x-1)(x-2))^2}$, and $f(0)=0$. Find $f(3)$.

Solution:

By partial fractions we find

$$\frac{x^2+2}{((x-1)(x-2))^2} = \frac{8}{x-1} + \frac{3}{(x-1)^2} - \frac{8}{x-2} + \frac{6}{(x-2)^2}.$$

Integrating this we find $f(x) = 8\ln|x-1| - \dfrac{3}{x-1} - 8\ln|x-2| - \dfrac{6}{x-2} + C.$

$f(0)=0$ gives $C = 8\ln 2 - 6$, and we find $f(3) = 16\ln 2 - \dfrac{27}{2}.$

Q-5) Evaluate the integral $\int x^2 \ln^2 x \, dx$.

Solution: We apply by-parts with $u = \ln^2 x$ to obtain:

$$\int x^2 \ln^2 x \, dx = \frac{1}{3}x^3 \ln^2 x - \frac{2}{3} \int x^2 \ln x \, dx.$$

This last integral is again attacked with by-parts where we take $u = \ln x$:

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

Putting these together we find:

$$\int x^2 \ln^2 x \, dx = \frac{1}{3}x^3 \ln^2 x - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C.$$
