Due Date: 8 December 2010 Wednesday NAME:..

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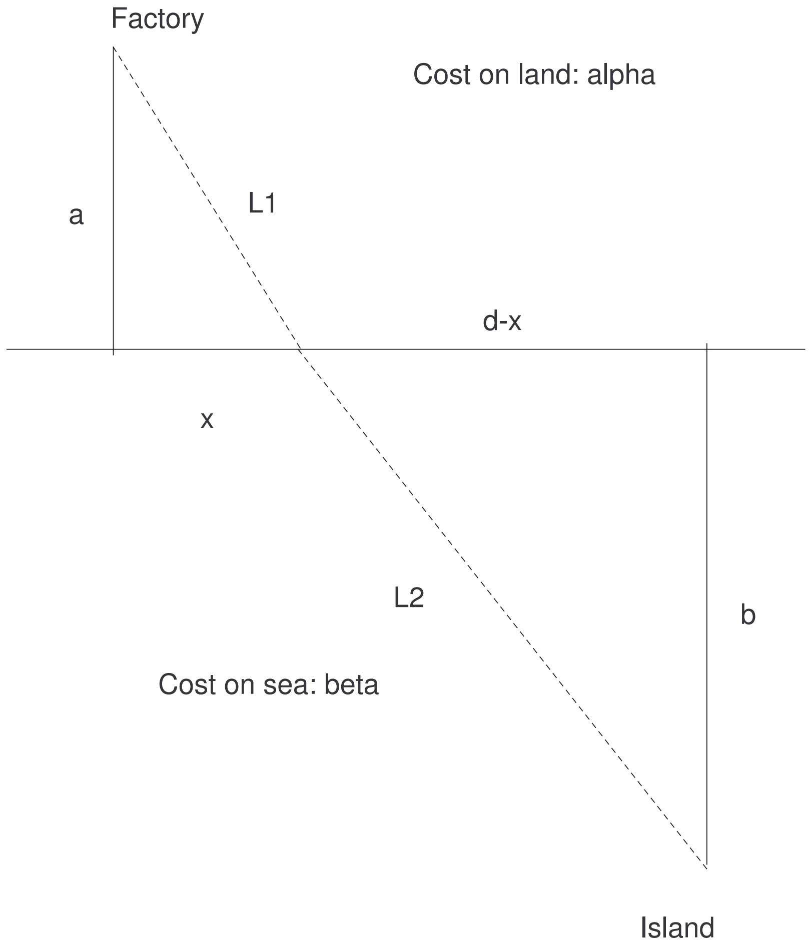
Math 113 Calculus – Homework 3

Question: We have a factory to which we want to provide coal from an island. The factory is located 50km inland from the coast line which is in the form of a straight line. The island is located 70 km out in the sea. The feet of the perpendiculars from the island and the factory on the shore line are 200km apart. We want to build a port on the shoreline so that the coal from the mine is first transported to this port and then carried to the factory. The cost of transportation on land is twice as much as that on sea. Where must we build this port so that we minimize our cost of transportation?

Show your work in reasonable detail.

Solution:



The total cost, using the variables in the figure is given by the function

$$f(x) = \alpha \cdot L_1 + \beta \cdot L_2, \quad x \in [0, d].$$

Here $L_1 = \sqrt{a^2 + x^2}$ and $L_2 = \sqrt{b^2 + (d-x)^2}$. Our function then becomes

$$f(x) = \alpha\sqrt{a^2 + x^2} + \beta\sqrt{b^2 + (d-x)^2}, \quad x \in [0, d].$$

To minimize this function we take its derivative and equate to zero.

$$f'(x) = \frac{\alpha x}{\sqrt{a^2 + x^2}} - \frac{\beta(d-x)}{\sqrt{b^2 + (d-x)^2}} = 0.$$

To solve this by hand, we need to do cross-multiplication and take squares. This eventually gives a quartic equation which is impractical to solve by hand.

$$(\alpha^2-\beta^2)x^4-2d(\alpha^2-\beta^2)x^3+[\alpha^2(b^2+d^2)-\beta^2(a^2+d^2)]x^2+(2a^2d\beta^2)x-a^2d^2\beta^2=0.$$

We observe that this equation will reduce to a second degree equation if (i) $\alpha=\beta$, or (ii) $a=0$ and $\alpha=0$, or (iii) $b=0$ and $\beta=0$. If we substitue the values of the problem into this function, we get

$$3x^4-1200x^3+137100x^2+1000000x-100000000=0.$$

Its roots are -27.19, 26.17, $200-81i$ and $200+81i$. The only relevant root for us is the real one in the interval $[0,200]$. So we take the root 26.17.

However we can do better. Put in the given values of the problem

$$a=50,\ b=70,\ d=200,\ \alpha=2,\ \beta=1.$$

into the function f from the beginning to get

$$f(x)=2\sqrt{2500+x^2}+\sqrt{44900-400\,x+x^2},\ x\in[0,200],$$

and

$$f'(x)=\frac{2x}{\sqrt{2500+x^2}}+\frac{x-200}{\sqrt{44900-400\,x+x^2}}=0.$$

Now use a computer algebra program to solve this. The solution is

$$x=26.17589982.$$

Calculus tells us that the minimum occurs either at this point or at the end points. So we calculate:

$$f(0)=311,\ f(26.17)=300,\ f(200)=482.$$

Hence the minimum occurs at $x=26.17589982$.

Therefore the port must be built 26km 175m away from the feet of the perpendicular from the factory on the coast line, towards the island side of course.