Due Date: October 19, 2011 Wednesday NAME:..

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Math 113 Calculus – Homework 1 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are 4 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Consider the function $f(x) = \dfrac{1}{x}$ for $x > 0$.

For each given $\epsilon > 0$ and for each $x_0 > 0$, find explicitly a $\delta > 0$ (which usually depends both on ϵ and x_0) such that for all $x > 0$ with $|x - x_0| < \delta$ we will have $|f(x) - f(x_0)| < \epsilon$.

Solution:

Fix an $\epsilon > 0$ and any $x_0 > 0$. Choose a small $\delta > 0$ such that $x - \delta > 0$. We want to find out how small δ should be such that whenever we choose a point y with $|x_0 - y| < \delta$, we will have $|f(x_0) - f(y)| < \epsilon$.

Since $0 < x_0 - \delta < y < x_0 + \delta$, we have $\dfrac{1}{x_0 + \delta} < \dfrac{1}{y} < \dfrac{1}{x_0 - y}$.

Now we examine how we can make $f(y)$ close to $f(x_0)$.

$$|f(y) - f(x_0)| = \frac{|x_0 - y|}{x_0 y} < \frac{\delta}{x_0(x_0 - \delta)}.$$

We now force this last expression to be smaller than or equal to ϵ to find a condition on δ. This gives

$$0 < \delta \leq \frac{\epsilon\, x_0^2}{1 + \epsilon\, x_0}.$$

Hence any δ satisfying this inequality will work.

Observe that, once $\epsilon > 0$ is fixed, the values of δ depends on x_0. In particular, if we consider $\delta = \delta(x, \epsilon)$ as a function of x and ϵ, then for any $\epsilon > 0$, we have $\lim\limits_{x \to 0^+} \delta(x, \epsilon) = 0$.

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Q-2) Consider the function $f(x) = \dfrac{1}{x}$ for $x > 0$.

Prove or disprove that given any $\epsilon > 0$, there exists a $\delta > 0$ (which depends only on ϵ) such that for all $x, y > 0$ with $|x - y| < \delta$ we will have $|f(x) - f(y)| < \epsilon$.

Solution:

We will disprove the given statement.

Thus, what we will prove is the following statement: There is an $\epsilon > 0$ such that for every $\delta > 0$, there exists $x, y > 0$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

Indeed take $\epsilon = 1$. For any $\delta > 0$, take $y = x + \delta/2$ where we want to choose $x > 0$ such that

$$|f(x) - f(y)| = \frac{|x - y|}{xy} = \frac{\delta/2}{x(x + \delta/2)} \geq 1.$$

It is possible to choose such an $x > 0$ since the limit of the last expression as x goes to zero from the right is infinite.

Technically speaking, we proved that $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$.

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Q-3) Consider the function $f(x) = \dfrac{1}{x}$ for $x \in [1, 5]$.

Prove or disprove that given any $\epsilon > 0$, there exists a $\delta > 0$ (which depends only on ϵ) such that for all $x, y \in [1, 5]$ with $|x - y| < \delta$ we will have $|f(x) - f(y)| < \epsilon$.

Solution:

First observe that if we can find a $\delta > 0$ which works for a particular ϵ, then the same δ works for larger ϵ also. So there is no harm in assuming that $0 < \epsilon < 1$.

In question 1 we showed that for any $\epsilon > 0$ and any $x > 0$, the choice of $\delta = \dfrac{\epsilon x^2}{1 + \epsilon x}$ works. This shows that once ϵ is fixed, δ depends continuously on x.

Since $x \in [1, 5]$ which is a closed and bounded interval, this continuous function has a minimum value there.

Since $\delta = \delta(x) = \dfrac{\epsilon x^2}{1 + \epsilon x} > \epsilon > 0$, the minimum of $\delta(x)$ on $[1, 5]$ is strictly positive (in fact larger than or equal to ϵ). Call this minimum value δ_0.

We showed that $\delta_0 > 0$ and works for the given ϵ at every point of the interval $[1, 5]$.

You notice that the only property of the interval $[1, 5]$ we used is its being closed and bounded. Hence $1/x$ is uniformly continuous on every closed and bounded subinterval of positive real numbers.

This is a consequence of the theorem which we proved and which says that a continuous function on a closed and bounded interval is uniformly continuous.

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Q-4) Consider the function $f(x) = \frac{1}{x}$ for $x \in [1, \infty)$.

Prove or disprove that given any $\epsilon > 0$, there exists a $\delta > 0$ (which depends only on ϵ) such that for all $x, y \in [1, \infty)$ with $|x - y| < \delta$ we will have $|f(x) - f(y)| < \epsilon$.

Solution:

In the previous problem, we proved that $1/x$ is uniformly continuous on every interval of the form $[1, R]$ for any $R > 1$. We need to control $1/x$ on $[R, \infty)$. But this is easy since $1/x$ tends to zero for large x. We exploit this observation to show that $1/x$ is uniformly continuous on $[1, \infty)$.

Fix any $\epsilon > 0$. Choose $R > 0$ such that for any $x \geq R$, we will have $1/x < \epsilon/3$.

Since $1/x$ is uniformly continuous on $[1, R]$, feeding $\epsilon/3 > 0$ into the definition of uniform continuity of $1/x$ on $[1, R]$, we find that there exists a $\delta > 0$ such that

$$\left| \frac{1}{x} - \frac{1}{y} \right| < \epsilon/3 \quad \text{for all } x, y \in [1, R] \text{ such that } |x - y| < \delta.$$

Now take any $x, y \in [1, \infty)$ with $|x - y| < \delta$. We have three cases:

Case 1: $x, y \in [1, R]$. We just showed that in this case $|f(x) - f(y)| < \epsilon/3 < \epsilon$.

Case 2: $x, y \in [R, \infty)$. By choice of R, we have

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon/3 + \epsilon/3 < \epsilon.$$

Case 3: $0 < x < R < y$. In this case, we use the conclusions of the previous two cases as follows.

$$|f(x) - f(y)| \leq |f(x) - f(R)| + |f(R) - f(y)| < \epsilon/3 + 2\epsilon/3 = \epsilon.$$

Thus we showed that $1/x$ is uniformly continuous on $[1, \infty)$.

Note: There is an easier way to capture a working δ, which I learned from several student solutions. This works both for questions 3 and 4.

Since in both cases $x, y \geq 1$, we have $\frac{1}{xy} \leq 1$. Then we have for all $x, y \geq 1$,

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} \leq |x - y|.$$

If we restrict this last expression to be less than ϵ, then that will surely restrict $|f(x) - f(y)|$. But this last expression is already less than δ. So any δ with $0 < \delta \leq \epsilon$ will work, for both problems.