Due Date: December 7, 2011 Wednesday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 113 Calculus – Homework 3 – Solutions

1	2	TOTAL
25	75	100

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. No hanging expressions will be read. No interpretation of your work will be attempted. Write everything you want us to know. Do not leave anything to mind reading; we do not do that in Math department, yet!

Q-1) Show that e is not a rational number.

> **Hint:** *Use Taylor's Theorem to approximate e. You can use that $e < 3$, which we proved in class.*

Solution:

Assume that $e = \dfrac{M}{N}$, where M and N are positive integers.

Choose n such that $n + 1 = 3NK$ for some positive integer K.

By Taylor's Theorem, we have

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c}{(n+1)!} x^{n+1}, \quad \text{for some } c \text{ between } x \text{ and } 0.$$

Putting in $x = 1$, we have

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^c}{(n+1)!}, \quad \text{for some } c \text{ between } 1 \text{ and } 0.$$

Since $0 < e < 3$, we have for the error term

$$0 < \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!}.$$

This leads to

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} = \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!},$$

from which we can write

$$0 < \frac{M}{N} - \sum_{k=0}^{n} \frac{1}{k!} < \frac{3}{(n+1)!}.$$

Multiply all sides by $(n+1)!$ to obtain

$$0 < 3MK\, n! - \sum_{k=0}^{n} [(k+1)(k+2)\cdots(n+1)] < 3,$$

or

$$0 < 3MK\, n! - 3NK[1 + n + n(n-1) + \cdots + n(n-1)\cdots 1] < 3.$$

Dividing all sides by 3, we get

$$0 < MK\, n! - NK[1 + n + n(n-1) + \cdots + n(n-1)\cdots 1] < 1.$$

This is an inequality about an integer which lies strictly between 0 and 1. This is clearly a contradiction which arose from our assumption that e is rational.

Hence e is not rational.

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Q-2)

(Designing a dustpan) Equal squares are cut out of two adjacent corners of a square of sheet metal having sides of length 25 cm. The three resulting flaps are bent up, as shown in Figure 4.74, to form the sides of a dustpan. Find the maximum volume of a dustpan made in this way.

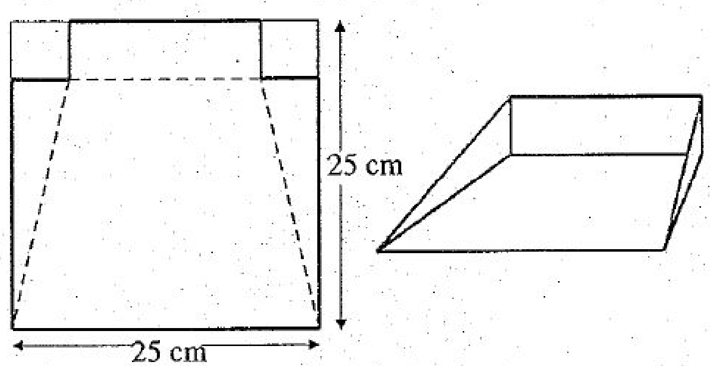
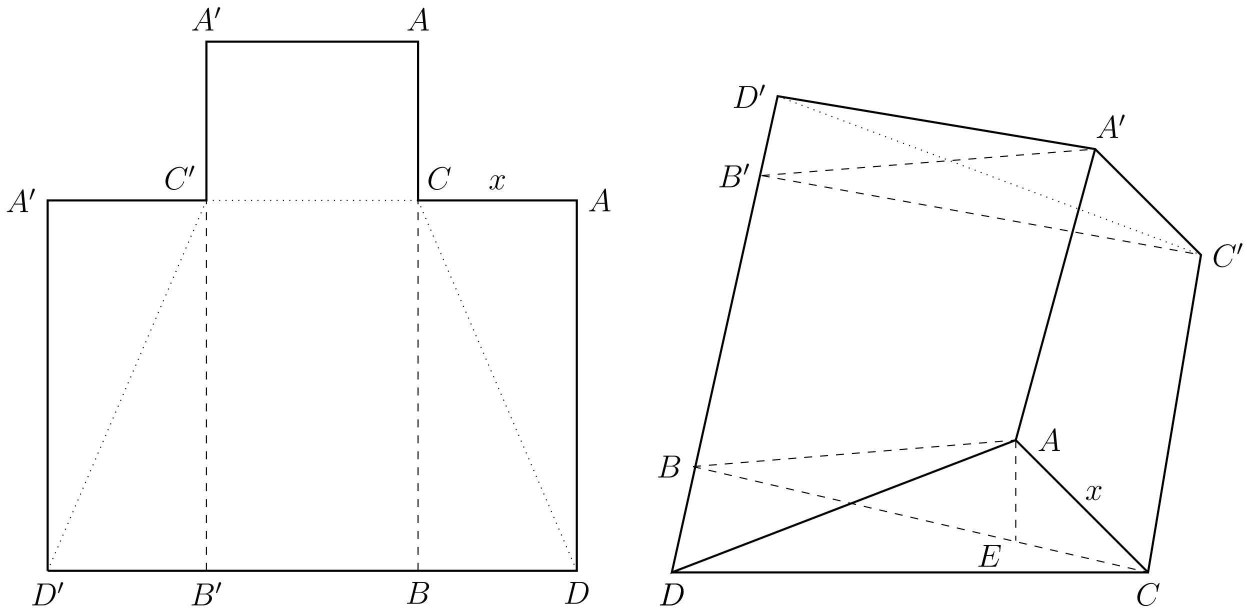


Figure 4.74

Solution on next page:



(a) Before folding

(b) After folding

Figure 1: Dustpan from a square metal sheet.

The lengths of AC and $A'C'$ are x. The total length of one side $DD' = AB$ is taken to be L cm for a general solution. After we solve the problem we can put back $L = 25$.

In the figure AE is perpendicular to the plane $CDD'C'$. In particular it is perpendicular to BC. Moreover CB is perpendicular to DD'.

The triangle ΔABD is a right triangle with hypothenuse AD. Since from construction $|AD| = L - x$ and $|BD| = x$, we find that $|AB| = \sqrt{(L-x)^2 - x^2}$.

In ΔABC, we know the lengths of the sides. $|AB| = \sqrt{(L-x)^2 - x^2}$, $|AC| = x$ and $|CB| = L - x$. We notice that $|AB|^2 + |AC|^2 = |CB|^2$, so $BA \perp AC$.

Let AE be the perpendicular to BC from A. Denote the length of AE by h.

Since $\Delta ABC \sim \Delta EAC$, we have

$$\frac{|AB|}{|BC|} = \frac{h}{x},$$

from where we solve for h.

$$h = \frac{x}{L-x}\sqrt{(L-x)^2 - x^2}.$$

Let V_1 be the volume of the pyramid $ABDC$. Then V_1 is $h/3$ times the area of ΔBCD. Hence

$$V_1 = \frac{x^2}{6}\sqrt{(L-x)^2 - x^2}.$$

Notice that the volume of the pyramid $A'B'D'C'$ is also V_1.

Let V_2 be the volume of the triangular prism with base ΔABC and height $|AA'| = x$. Thus

$$V_2 = \frac{1}{2}x(L-2x)\sqrt{(L-x)^2 - x^2}.$$

Finally we find that the total volume $V = 2V_1 + V_2$ is given by

$$V(x) = x\sqrt{(L-x)^2 - x^2}\left(\frac{L}{2} - \frac{2}{3}x\right), \quad \text{for} \quad 0 \leq x \leq \frac{L}{2}.$$

Notice that $V(0) = V(L/2) = 0$, so the maximum value will occur at an interior critical point.

$$V'(x) = \frac{L}{6}\frac{(4x-L)(5x-3L)}{\sqrt{L^2 - 2Lx}}.$$

$V'(x) = 0$ holds when $x = \dfrac{L}{4}$ which is an interior point. The other critical point, $3L/2$ is not in the interval $[0, L/2]$. Hence $x = L/4$ must give the maximum value, since V is always positive on the interior and zero at the boundary.

$$V\left(\frac{L}{4}\right) = \frac{\sqrt{2}}{24}L^3 \approx 0.0589L^3.$$

When $L = 25$, the critical point is $x = 25/4$, and the maximal volume is $920.71\ \text{cm}^3$.