Due Date: January 6, 2012 Friday NAME:..

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Math 113 Calculus – Homework 4 – Solution

1	TOTAL
100	100

Please do not write anything inside the above boxes!

Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. No hanging expressions will be read. No interpretation of your work will be attempted. Write everything you want us to know. Do not leave anything to mind reading; we do not do that in Math department, yet!

Q-1)

A martini glass in the shape of a right-circular cone of height h and semivertical angle α (see Figure 7.14) is filled with liquid. Slowly a ball is lowered into the glass, displacing liquid and causing it to overflow. Find the radius R of the ball that causes the greatest volume of liquid to overflow out of the glass.

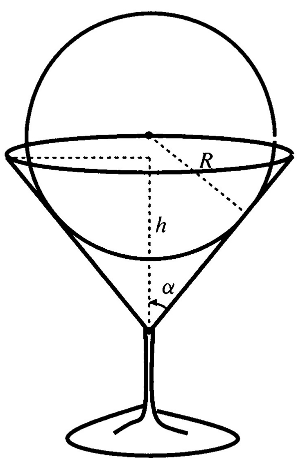
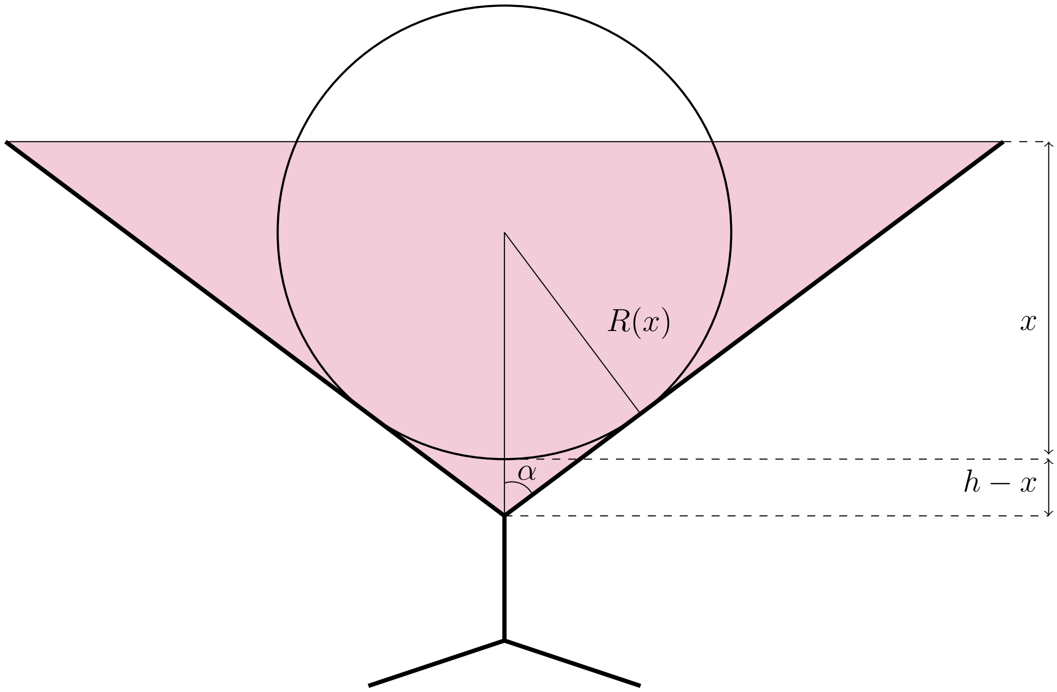


Figure 7.14

Solution on next page.



The volume of the sphere inside the liquid is given by

$$V(x) = \frac{\pi x^2}{3}(3R(x) - x),$$

where $0 \le x \le 2R(x)$. When $x > R(x)$, the sphere is totally inside the liquid and the its radius is decreasing, and will not contribute to the maxima. Moreover, as x decreases, the point of tangency of the ball with the sides of the glass will move up and we must stop at the point where the ball is tangent to the glass at the brim of the glass. From there on the radius of the ball may increase but the amount of liquid it displaces from the glass will be less than the amount that the ball tangent to the glass at the brim displaces. These considerations dictate the upper and lower bounds of x in the formula. Therefore we first address the problem of determining these bounds.

Throughout these calculations we take $h > 0$, and also $0 < \alpha < \pi/2$ so that $0 < \sin\alpha < 1$.

From the figure, we observe that

$$\sin\alpha = \frac{R(x)}{R(x) + h - x}.$$

It follows that

$$R(x) = \frac{(h-x)\sin\alpha}{1-\sin\alpha}, \quad \text{and} \quad R'(x) = -\frac{\sin\alpha}{1-\sin\alpha}.$$

If we place the center of the circle above liquid level, we still get the same relation between x and $R(x)$.

When the ball is tangent to the side of the glass at the brim, we find that $R(x) = h\sec\alpha\tan\alpha$. The corresponding value of x is the smallest value of x which we want to consider. Solving

$$\frac{(h-x)\sin\alpha}{1-\sin\alpha} = h\sec\alpha\tan\alpha$$

for x we find

$$x_{\min} = \frac{h\sin\alpha}{1+\sin\alpha}.$$

The largest value of x is when $x = 2R(x)$. Solving this for x gives

$$x_{\max} = \frac{2h\sin\alpha}{1+\sin\alpha}.$$

Finally, our problem is to find the value of x which maximizes

$$V(x) = \frac{\pi x^2}{3}(3R(x) - x), \quad \text{for} \quad \frac{h\sin\alpha}{1+\sin\alpha} \le x \le \frac{2h\sin\alpha}{1+\sin\alpha},$$

and to find the corresponding value of $R(x)$.

We find that

$$V'(x) = \frac{\pi}{1-\sin\alpha}(2xh\sin\alpha - x^2(1+2\sin\alpha)),$$

and

$$V'(x) = 0 \quad \text{when} \quad x = 0 \quad \text{or} \quad x = \frac{2h\sin\alpha}{1+2\sin\alpha}.$$

The first root $x = 0$ is outside our range, but the second root satisfies

$$\frac{h\sin\alpha}{1+\sin\alpha} < \frac{2h\sin\alpha}{1+2\sin\alpha} < \frac{2h\sin\alpha}{1+\sin\alpha},$$

so it is a critical point. We also see that

$$V''(x) = \frac{\pi}{1-\sin\alpha}(2h\sin\alpha - 2x(1+2\sin\alpha)),$$

and

$$V''(\frac{2h\sin\alpha}{1+2\sin\alpha}) = -\frac{2\pi\sin\alpha}{1-\sin\alpha} < 0,$$

so this critical point gives the maximum values of the function. Corresponding to this critical point, we find

$$R(\frac{2h\sin\alpha}{1+2\sin\alpha}) = \frac{h\sin\alpha}{\sin\alpha + \cos 2\alpha}.$$

We can further investigate the position of the center of the ball when maximal displacement holds. This position of course depends on the value of α.

When the center of a ball is at the liquid level, its radius satisfies $R = h\sin\alpha$. When

$$R(\frac{2h\sin\alpha}{1+2\sin\alpha}) = \frac{h\sin\alpha}{\sin\alpha + \cos 2\alpha} < h\sin\alpha,$$

the center of the ball which causes maximal displacement is below liquid level. Solving for the corresponding α, we find that if $\alpha < \pi/6$, the center of the ball which causes maximal displacement is below liquid level. If $\alpha = \pi/6$, it is at liquid level, and if $\alpha > \pi/6$, it is above liquid level.

Finally we mention a curious fact. Corresponding to any $0 < \alpha < \pi/2$, let $C(\alpha)$ denote the volume of the glass. Then

$$C(\alpha) = \pi h^3 \tan^2\alpha.$$

The ratio of maximal displacement to the volume is given by

$$\frac{V(\frac{2h\sin\alpha}{1+2\sin\alpha})}{C(\alpha)} = \frac{4}{3}\,\frac{\sin\alpha + \sin^2\alpha}{1+4(\sin\alpha+\sin^2\alpha)},$$

and

$$\lim_{\alpha\to(\pi/2)^-} \frac{V(\frac{2h\sin\alpha}{1+2\sin\alpha})}{C(\alpha)} = \frac{8}{27}.$$