MATH 114 HOMEWORK 4 SOLUTIONS

1. pg.976-24: Find local extrema.
$f(x+y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$. Hence, $f_x = 6x^2 - 18x = 0$, $f_y = 6y^2 + 6y - 12 = 0$.

Then, $x = 0, x = 3$ and $y = -2, y = 1$. We will look at Δ to find local max,local min and saddle points. As $\Delta = f_{xx}f_{yy} - f_{xy}^2$ and

$$f_{xx} = 12x - 18 \ , \ f_{yy} = 12y + 6 \ \text{ and } \ f_{xy} = 0$$

we have, $\Delta < 0$ at the points $(3, -2)$ and $(0, 1)$. So these are the saddle points.

We have $\Delta > 0$ at the points $(3, 1)$ and $(0, -2)$. As $f_{xx} > 0$ at $(3, 1)$ it is local minimum and as $f_{xx} < 0$ at $(0, -2)$ it is local maximum.

2. pg.1010-34: Integrate:

$$\begin{aligned}
\int_0^2 \int_0^{4-x^2} \left(\frac{xe^{2y}}{4-y}\right) dy\, dx &= \int_0^4 \int_0^{\sqrt{4-y}} \left(\frac{xe^{2y}}{4-y}\right) dx\, dy \\
&= \int_0^4 \left(\frac{x^2e^{2y}}{2(4-y)}\right) |_0^{\sqrt{4-y}} dy \\
&= \frac{e^8 - 1}{4}.
\end{aligned}$$

3. pg.1010-15: Integrate $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$.

$$\begin{aligned}
\int_0^1 \int_0^{1-u} (v - \sqrt{u}) dv\, du &= \int_0^1 \left(\frac{v^2}{2} - \sqrt{u}\, v\right) |_0^{1-u} du \\
&= \frac{-1}{10}.
\end{aligned}$$

Equivalently,

$$\int_0^1 \int_0^{1-v} (v - \sqrt{u}) du\, dv = \frac{-1}{10}.$$

4. pg.976-38: Find absolute extrema on the triangular plate bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the first quadrant.
$f(x, y) = 4x - 8xy + 2y + 1$. Hence, $f_x = 4 - 8y$, $f_y = 2 - 8x$,

i. on $y = 0$: We have $f(x, y) = f(x, 0) = 4x+1$, $0 \leq x \leq 1$. The possible extreme values at the endpoints are

$$f(0, 0) = 1 \text{ and } f(1, 0) = 5$$

The interior ones must satisfy $f'(x, 0) = 0$ but we have $f'(x, 0) = 4 > 0$. So we do not have an extreme value in the interior.

ii. on $x = 0$: We have $f(x, y) = f(0, y) = 2y + 1$, $0 \leq y \leq 1$. The possible extreme values at the endpoints are

$$f(0, 0) = 1 \text{ and } f(1, 0) = 3$$

The interior ones must satisfy $f'(0, y) = 0$ but we have $f'(0, y) = 2 > 0$. So we do not have an extreme value in the interior.

iii. on $x + y = 1$: We have $f(x, y) = f(x, 1 - x) = 8x^2 - 6x + 3$. Hence, $f'(x, 1 - x) = 0$ gives $x = 3/8$. So, $y = 5/8$ and the possible extreme value is $f(x, y) = f(3/8, 5/8) = 15/8$.

iv. We should also look at the points satisfying $f_x = f_y = 0$. In our case this is the point $(1/4, 1/2)$ and it gives us the value 2.

Hence we have the values $1, 2, 3, 5, 15/8$. Clear that, the maximum is 5 attained at $(0, 1)$ and the minimum is 1 attained at $(0, 0)$.

4. pg.987-16: Design a container in the shape of a cylindrical tank with hemispherical ends. Its volume must be 8000 m^3. What should the dimensions be so that the least amount of surface material will be needed?

$V = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$. So let, $f(r, h) = \frac{4}{3}\pi r^3 + \pi r^2 h - 8000$ and $g(r, h) = f_r = 4\pi r^2 + 2\pi rh$. Then, by $\nabla f = \lambda \nabla g$ we get

$$(8\pi r + 2\pi h)\hat{\imath} + (2\pi r)\hat{\jmath} = \lambda \left[(4\pi r^2 + 2\pi rh)\hat{\imath} + (\pi r^2)\hat{\jmath}\right].$$

So, $r = \frac{2}{\lambda}$, where λ is nonzero. In fact if $\lambda = 0$ then $V = 0$ will hold which is nonsense. So we have $r = \frac{2}{\lambda}$, implying $h = 0$. Substituting in f we get, $r = 10\sqrt[3]{\frac{6}{\pi}}$.

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