Due on February 13, 2006, Monday

MATH 114 Homework 1 – Solutions

1. [p631-ex65] Find the values of p for which each integral converges.

$$\textbf{a}. \int_1^2 \frac{dx}{x(\ln x)^p}, \qquad \textbf{b}. \int_2^\infty \frac{dx}{x(\ln x)^p}.$$

Solution-a: Changing variables by $u = \ln x$ converts the integral to $\int_0^{\ln 2} \frac{dx}{x(\ln x)^p}$. When $p = 1$, this is $\left(\ln u|_0^{\ln 2}\right)$ which diverges since ln is not defined at 0. When $p \neq 1$, the integral evaluates to $\left(\frac{u^{1-p}}{1-p}|_0^{\ln 2}\right)$ which converges if and only if $1 - p > 0$, to avoid division by zero. So this integral converges for $p < 1$ and diverges for $p \geq 1$.

Solution-b: The same change of variables now converts the integral into $\int_{\ln 2}^\infty \frac{dx}{x(\ln x)^p}$. Again for $p = 1$ we have divergence since $\ln u$ goes to infinity as u goes to infinity. When $p \neq 1$ we now need to have $1 - p < 0$ so that when u goes to infinity the value of the integral converges. Hence this integral converges when $p > 1$ and diverges when $p \leq 1$.

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2. [p637-ex144] Evaluate the following improper integral.

$$\int_{-\infty}^\infty \frac{4dx}{x^2 + 16}.$$

Solution:

$$\begin{aligned}
\int_{-\infty}^\infty \frac{4dx}{x^2+16} &= \lim_{R\to\infty} \int_{-R}^0 \frac{4dx}{x^2+16} + \lim_{R\to\infty} \int_0^R \frac{4dx}{x^2+16} \\
&= \lim_{R\to\infty} \arctan(x/4)|_{-R}^0 + \lim_{R\to\infty} \arctan(x/4)|_0^R \\
&= (0 - \frac{-\pi}{2}) + (\frac{\pi}{2} - 0) \\
&= \pi.
\end{aligned}$$

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3. [**p637-ex148**] Does this improper integral converge or diverge?

$$\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt.$$

Solution: $\dfrac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$.

$\displaystyle\int_1^\infty e^{-t}\,dt$ converges, by direct computation.

So the given integral converges by comparison.

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4. [**p638-ex218**] Evaluate the integral

$$\int_2^\infty \frac{4v^3+v-1}{v^2(v-1)(v^2+1)}\,dv.$$

Solution:

$$\begin{aligned}
\int_2^\infty \frac{4v^3+v-1}{v^2(v-1)(v^2+1)}\,dv &= \lim_{R\to\infty}\int_2^R \frac{4v^3+v-1}{v^2(v-1)(v^2+1)}\,dv \\
&= \lim_{R\to\infty}\int_2^R \left(\frac{1-2v}{v^2+1}+\frac{1}{v^2}+\frac{2}{v-1}\right)\,dv \\
&= \lim_{R\to\infty}\left(\arctan v - \ln(v^2+1) - \frac{1}{v} + 2\ln(v-1)\Big|_2^R\right) \\
&= \lim_{R\to\infty}\left(\ln\frac{(v-1)^2}{v^2+1} + \arctan v - \frac{1}{v}\Big|_2^R\right) \\
&= \frac{\pi}{2} - \arctan 2 + \ln 5 + \frac{1}{2}.
\end{aligned}$$

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5. [**p638-ex220**] Find a positive number a satisfying

$$\int_0^a \frac{dx}{1+x^2} = \int_a^\infty \frac{dx}{1+x^2}.$$

Solution: Evaluating both sides (the antiderivative is $\arctan x$), we find $\arctan a - 0 = \dfrac{\pi}{2} - \arctan a$. Hence $\arctan a = \dfrac{\pi}{4}$ and $a = \tan\frac{\pi}{4} = 1$.
