Due on May 9, 2006, Tuesday, Class time. No late submissions!

MATH 114 Homework 9 – The last one :-)
Solutions

1: Intersect the sphere $x^2 + y^2 + z^2 = 196$ with the cylindrical surface $x^2 + y^2 = 14y$, $z \geq 0$, and calculate **(i)** the area of the spherical cap so formed and **(ii)** the volume under this cap and over the xy-plane.

Solution (i): Let $f(x, y, z) = x^2+y^2+z^2-196$. Then $|\nabla f| = 28$ and $|\nabla f \cdot \mathbf{k}| = 2z$, and $d\sigma = \dfrac{14}{z}$. Since the denominator vanishes at a certain point in the domain, our integral should not pass that point.

The disk bounded by the circle $x^2 + y^2 = 14y$ is represented by $r = 14\sin\theta$, $0 \leq \theta \leq \pi$ in polar coordinates. Let R denote the part of this disc lying in the first quadrant of the xy-plane. Since $z = 0$ for $\theta = \pi/2$, this is the right region of integration.

$$\begin{aligned}
\frac{1}{2}\text{Area} = \int\int_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dxdy &= \int\int_R \frac{14}{z}\, dxdy \\
&= 14 \int\int_R \frac{1}{\sqrt{196 - x^2 - y^2}}\, dxdy = 14\int_0^{\pi/2}\int_0^{14\sin\theta} \frac{r\, dr\, d\theta}{\sqrt{196 - r^2}} \\
&= 14 \int_0^{\pi/2} \left(-\sqrt{196 - r^2}\,\Big|_0^{14\sin\theta} \right) d\theta \\
&= 14 \int_0^{\pi/2} (14 - 14\cos\theta)\, d\theta = 14\left(14\theta - 14\sin\theta\Big|_0^{\pi/2}\right) \\
&= 98\pi - 196. \\
\text{Area} &= 196\pi - 392 \approx 223.75.
\end{aligned}$$

Solution (ii):

$$\text{Volume} = 2\int\int_R z\,\frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dxdy = 28\int\int dxdy = 28(\text{ Area of } R\ ) = 28(49\pi/2) = 686\pi.$$

2: Find the area of that portion of the sphere $x^2 + y^2 + z^2 = 4$ lying between the planes $z = \sqrt{3}$ and $z = -1$.

Solution: We can parameterize this region by
$r(\phi, \theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, $0 \le \theta \le 2\pi$, $\pi/6 \le \phi \le 2\pi/3$. Then

$$\text{Area} = \int_0^{2\pi}\int_0^{2\pi/3} |r_\phi \times r_\theta|\, d\phi d\theta = \int_0^{2\pi}\int_0^{2\pi/3} 4\sin\phi\, d\phi d\theta = 4(1+\sqrt{3})\pi.$$

3: Find an equation for the plane through the origin such that the circulation of the flow $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ around the circle C of intersection of the plane with the sphere $x^2 + y^2 + z^2 = 4$ is a maximum. Recall that the circulation of the flow $\mathbf{F}$ around the circle C is given by $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r}$ is a smooth parametrization of C. What happens if we replace the radius of the sphere by some other value?

Solution: Let $Ax + By + Cz = 0$ be an equation of that plane. Assume without loss of generality that (A, B, C) is a unit vector, i.e. $A^2 + B^2 + C^2 = 1$. Change the circulation integral along the given circle to a curl integral on the surface D of the disk bounded by this circle. The normal to that disk is $n = (A, B, C)$.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_D \textbf{curl } \mathbf{F} \cdot n\, d\sigma = \int\int_D (1,1,1)\cdot(A,B,C)\, d\sigma \\ &= (A+B+C)\int\int_D d\sigma = (A+B+C)(4\pi)\end{aligned}$$

Maximize $A + B + C$ subject to the constraint $A^2 + B^2 + C^2 = 1$. This gives $A = B = C = 1/\sqrt{3}$. Therefore an equation of the desired plane is $x + y + z = 0$.

4: Calculate the area of the region on the Earth bounded by the meridians 120° and 150° west longitude and the circles 30° and 45° north latitude, assuming that the Earth is spherical with radius R km.

Solution: By rotating the earth a little bit(!) we can assume that the parametrization of the required region is given by $\mathbf{r} = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$, where $0 \le \theta \le \pi/6$ and $0 \le \phi \le \pi/12$.

$$\begin{aligned}\text{Area} &= \int_0^{\pi/6}\int_0^{\pi/12} |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\phi d\theta \\ &= R^2\int_0^{\pi/6}\int_0^{\pi/12} \sin\phi\, d\phi d\theta = R^2\left(\theta\big|_0^{\pi/6}\right)\left(-\cos\phi\big|_0^{\pi/12}\right) = R^2(\pi/6)(1-\cos\pi/12),\end{aligned}$$

where $\cos \pi/12$ can be calculated from half angle formula to find

$$\text{Area} = R^2(\pi/6)(1 - (1/2)(\sqrt{3} + 2)^{(1/2)}) \approx (0,01784)R^2.$$

To convince yourself about the validity of this formula observe that the whole surface area of the sphere will be given by

$$\text{Area of sphere} = R^2 \left(\theta |_{0}^{2\pi} \right) \left(-\cos \phi |_{0}^{\pi} \right) = 4\pi R^2.$$

5: Find the outward flux of the vector field $\mathbf{F} = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$ across the boundary ∂D of the cube D cut from the first octant by the planes $x = 1$, $y = 1$ and $z = 1$. The outward flux of $\mathbf{F}$ on ∂D is given by the integral $\int \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\ d\sigma$, where $\mathbf{n}$ is the unit outward normal on the faces of ∂D.

Solution: Using divergence theorem, this integral becomes a triple integral on D.

$$\begin{aligned} \int \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\ d\sigma &= \int \int \int_{D} \nabla \cdot \mathbf{F}\ dV \\ &= 2 \int_0^1 \int_0^1 \int_0^1 (x + y + z)\ dx\ dy\ dz \\ &= 3. \end{aligned}$$

Send your comments to `sertoz@bilkent.edu.tr` please.