May 12, 2006, Friday, 10:40-11:30

Math 114 Calculus – Quiz II – Solutions

Q-1) Find the outward flux of the vector field $\mathbf{F} = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$ across the boundary ∂D of the cube D cut from the first octant by the planes $x = 1$, $y = 1$ and $z = 1$. The outward flux of $\mathbf{F}$ on ∂D is given by the integral $\int\int_{\partial D} \mathbf{F}\cdot\mathbf{n}\, d\sigma$, where $\mathbf{n}$ is the unit outward normal on the faces of ∂D.

Solution: Using divergence theorem, this integral becomes a triple integral on D.

$$\begin{aligned}\int\int_{\partial D} \mathbf{F}\cdot\mathbf{n}\, d\sigma &= \int\int\int_{D} \nabla\cdot\mathbf{F}\, dV \\ &= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\, dx\, dy\, dz \\ &= 3.\end{aligned}$$

Q-2) Evaluate the integral $\displaystyle\int\int_S xy\, d\sigma$, where S is the portion of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, lying over the triangle R in the xy-plane bounded by the lines $x = 0$, $x = y$ and $y = 1/\sqrt{2}$.

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2 - 1$.

$$\begin{aligned}\int\int_S xy\, d\sigma &= \int\int_R xy\frac{|\nabla f|}{|\nabla f\cdot\mathbf{k}|}\, dA \\ &= \int_0^{1/\sqrt{2}}\int_0^y \frac{xy}{\sqrt{1-x^2-y^2}}\, dxdy \\ &= -\int_0^{1/\sqrt{2}} y\left(\sqrt{1-x^2-y^2}\Big|_0^y\right)\, dy \\ &= -\int_0^{1/\sqrt{2}} y\left(\sqrt{1-2y^2}-\sqrt{1-y^2}\right)\, dy \\ &= \left(\frac{1}{6}(1-2y^2)^{3/2} - \frac{1}{3}(1-y^2)^{3/2}\Big|_0^{1/\sqrt{2}}\right) \\ &= -\frac{1}{6\sqrt{2}} + \frac{1}{6} \approx 0.05.\end{aligned}$$
