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Time: 10:35-12:35
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Math 114 Calculus – Midterm Exam II – Solutions

Q-1) Find the area of the triangle whose vertices are the points

$$P_0 = (7, 8, 9),\ P_1 = (8, 10, 12),\ P_2 = (11, 13, 15).$$

Solution:

Let $U = P_1 - P_0 = (1, 2, 3),\ V = P_2 - P_0 = (4, 5, 6)$. The area of the triangle is $\frac{1}{2}|U \times V|$.

We calculate $U \times V = (-3, 6, -3)$ and $|U \times V|^2 = 54$.

Hence the area is $\frac{\sqrt{54}}{2} \approx 3.67$.

Q-2) A curve in $\mathbb{R}^3$ is parameterized by the twice differentiable position vector $r(t)$ with $t \in \mathbb{R}$. Do **only one** of the following questions.

(i) Show that $\frac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$.

(ii) Show that $\ddot{r}(t)$ is orthogonal to $\mathbf{B}$.

(iii) If $\ddot{r}(t) = f(t)\, r(t)$ for some function $f : \mathbb{R} \to \mathbb{R}$, then show that $r(t)$ moves in a plane.

Solution:

(i) Thomas Calculus, 11th Edition, page 944.

(ii) Thomas Calculus, 11th Edition, page 945.

(iii) Thomas Calculus, 11th Edition, page 952.

Q-3) For each $\alpha \in \mathbb{R}$ define the function

$$f(x, y) = \begin{cases} \dfrac{x^\alpha y}{x^4 + y^6} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

For which values of α is the function f continuous at the origin?

Solution:

First and foremost observe that if $x = 0$ or $y = 0$, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ is zero. So we assume in the following arguments that $(x, y) \neq (0, 0)$.

In this problem the magical number is $10/3$. There are two ways to arrive at this number.

The first way is the ingenuous approach of Ali Adalı: Let $x_1 = \cdots = x_5 = x^4/5$ and $x_6 = y^6$. The Geometric Mean-Arithmetic Mean theorem states that

$$(x_1 \cdots x_6)^{1/6} \leq \frac{x_1 + \cdots + x_6}{6},$$

where the equality holds if and only if $x_1 = \cdots = x_6$, see Apostol's Calculus page 47 Ex 20. In our case this gives

$$c\,(x^4 + y^6) \geq x^{10/3}y, \text{ where } c = \frac{5^{5/6}}{6}.$$

Then we have

$$\left|\frac{x^\alpha y}{x^4 + y^6}\right| \leq c \left|\frac{x^\alpha y}{x^{10/3}y}\right| = c\,\left|x^{\alpha - 10/3}\right|$$

which goes to zero as $x \to 0$, if $\alpha > 10/3$. This shows that f is continuous at the origin for all $\alpha > 10/3$.

For $\alpha \leq 10/3$, let (x, y) approach $(0, 0)$ along the curve $y^3 = x^2$, i.e. $y = x^{2/3}$. Then we have

$$\left|\frac{x^\alpha y}{x^4 + y^6}\right| = (1/2)|x^{\alpha - 10/3}|.$$

This converges to $1/2$ as $x \to 0$ when $\alpha = 10/3$, and has no finite limit when $\alpha < 10/3$.

We conclude that f is continuous at the origin if and only if $\alpha > 10/3$.

The second way to arrive at the magical number $10/3$ is to experiment with the path $y^3 = x^2$ right from the beginning.

Q-4) A C^∞ function $f(x, y)$ on $\mathbb{R}^2$ is defined such that f and some of its derivatives take the following values at the given points.

	$(1,0)$	$(0,1)$	$(3,8)$	$(5,7)$	$(8,3)$	$(7,5)$
f	2	4	6	8	10	12
f_x	3	5	7	9	11	13
f_y	14	16	18	20	22	24
f_{xy}	15	17	19	21	23	25

Let $h(s, t) = f(5s + 3t, 7s + 8t)$. Find $h_s(0, 1)$ and $h_t(1, 0)$.

Solution:

$h_s(0, 1) = f_x(3, 8) \cdot 5 + f_y(3, 8) \cdot 7 = 7 \cdot 5 + 18 \cdot 7 = 161.$

$h_t(1, 0) = f_x(5, 7) \cdot 3 + f_y(5, 7) \cdot 8 = 9 \cdot 3 + 20 \cdot 8 = 187.$

Q-5) Among all rectangular boxes with open tops and of constant volume of 32 cubic units, find the dimensions of the ones with minimal and maximal surface area, if they exist.

Solution:

Let the the width, length and height of such a box be x, y, z, respectively. We know that $xyz = 32$ and since this is going to be a real box we must have $x, y, z > 0$.

The surface area is given by $xy + 2xz + 2yz$. Putting in $z = 32/(xy)$ we find

$$f(x,y) = xy + \frac{64}{y} + \frac{64}{x}, \; x, y > 0.$$

The critical point of this function is $(4, 4)$. The second derivative test gives this point as the local minimum point. Since this is the only critical point, then it must be the global minimum. Clearly f approaches infinity as x or y approaches to zero, so no maximal value exists.

The dimensions of the smallest surface value box are: width and length are 4 units and the height is 2 units.

Bonus:) Let $f(x, y, z)$ be a differentiable function with non-vanishing first derivatives. Show that if $f(x, y, z) = 0$, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

Solution:

Recall that $\left(\frac{\partial x}{\partial y}\right)_z$ means that y and z are free variables and x is a dependent value which is defined here by the relation $f(x, y, z) = 0$. Taking derivative of both sides of this equation with respect to y, using chain rule and keeping in mind that z is free, we get.

$f_x \cdot \left(\frac{\partial x}{\partial y}\right)_z + f_y = 0.$

Similarly we get $f_y \cdot \left(\frac{\partial y}{\partial z}\right)_x + f_z = 0$ and $f_x + f_z \cdot \left(\frac{\partial z}{\partial x}\right)_y = 0.$

The required identity now follows from these three equations.

Please forward any comments or questions to `sertoz@bilkent.edu.tr`