Due Date: May 2, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 114 Calculus – Homework 3 – Solutions

1	2	TOTAL
50	50	100

Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Let C be a piecewise smooth curve in the xy-plane that does not pass through the origin. Let $\theta = \theta(x, y)$ be the polar angle coordinate of the point $P = (x, y)$ on C, not restricted to an interval of length 2π, but varying continuously as P moves from one end of C to the other end.

(a) Show that $\nabla\theta = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$.

(b) Show that $\dfrac{1}{2\pi}\displaystyle\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$ is always an integer when C is a closed curve.

Solution:

If the Cartesian coordinates of a point are (x, y), then the θ of the polar coordinates is $\theta = \arctan(y/x)$. The gradient of θ is given by

$$\nabla\theta = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

If $\mathbf{r}$ is a parametrization of of the curve C, then

$$\frac{x\,dy - y\,dx}{x^2+y^2} = \nabla\theta \cdot d\mathbf{r}.$$

If further C is a closed curve, parametrized so that $\mathbf{r}(a) = \mathbf{r}(b)$, then $\theta(b) - \theta(a) = 2n\pi$, where $n \in \mathbb{Z}$ is the number of times C winds around the origin, $n \geq 0$ if the overall winding is counterclockwise, and $n < 0$ otherwise.

Finally, using the fundamental theorem of calculus, we find that

$$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi}\oint_C \nabla\theta \cdot d\mathbf{r} = \frac{1}{2\pi}\left(\theta\Big|_a^b\right) = n.$$

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Q-2) A smooth surface S is given parametrically by

$$\mathbf{r} = (\cos 2u)(2 + v\cos u)\mathbf{i} + (\sin 2u)(2 + v\cos u)\mathbf{j} + v\sin u\mathbf{k}$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Show that for *every* smooth vector field $\mathbf{F}$ on S,

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = 0,$$

where $\mathbf{N} = \mathbf{N}(u, v)$ is a unit normal vector field on S that depends continuously on (u, v).

Solution:

A normal vector to the surface is $\mathbf{r}_u \times \mathbf{r}_v$. But the parametrization satisfies $\mathbf{r}(u + \pi, v) = \mathbf{r}(u, -v)$. This is a Mobius band traced twice by the parametrization. Because of the above identity, the normal vector changes direction the second time around and cancels the effect of the previous pass. Hence the overall effect of $\mathbf{F}\cdot\mathbf{N}$ on$S$ is zero.