Due Date: February 22, 2012 Wednesday NAME:..

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Math 114 Calculus – Homework 1 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100
25	25	25	25	100

Please do not write anything inside the above boxes!

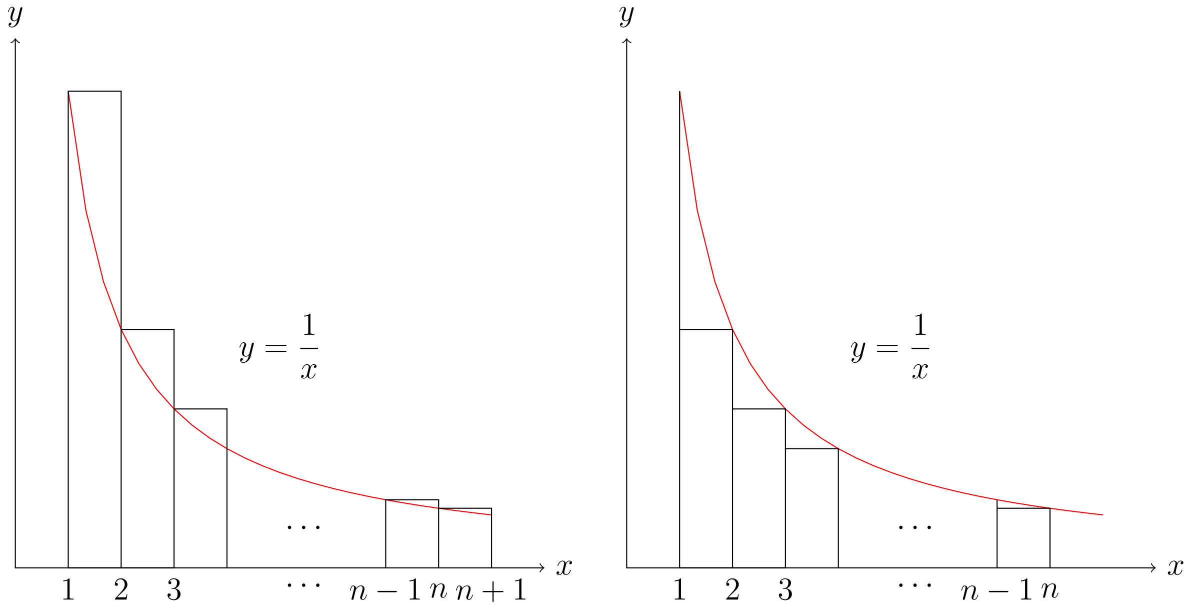
Check that there are 4 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1)

i. Let $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$. Find, if it exists, $\lim_{n\to\infty} a_n$.

ii. Let $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{n^2}$. Find, if it exists, $\lim_{n\to\infty} a_n$.

Solution on next page:



(a) Large boxes (b) Small boxes

Figure 1: Comparing areas of boxes with $\ln x$.

i. From Figure 1-a, we see that the sum of the areas of the boxes on the interval $[1, n+1]$ is larger than the total area under the curve $y = 1/x$ from $x = 1$ to $x = n+1$. This gives

$$\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

From Figure 1-b, we see that the sum of the areas of the boxes on the interval $[1, n]$ is smaller than the total area under the curve $y = 1/x$ from $x = 1$ to $x = n$. This gives

$$\frac{1}{2} + \cdots + \frac{1}{n} < \ln n.$$

Putting these two inequalities together, we find

$$\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n.$$

Subtracting $\ln n$ from all sides we find

$$0 < \ln(n+1) - \ln n < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n = a_n < 1,$$

which shows that the sequence a_n is bounded.

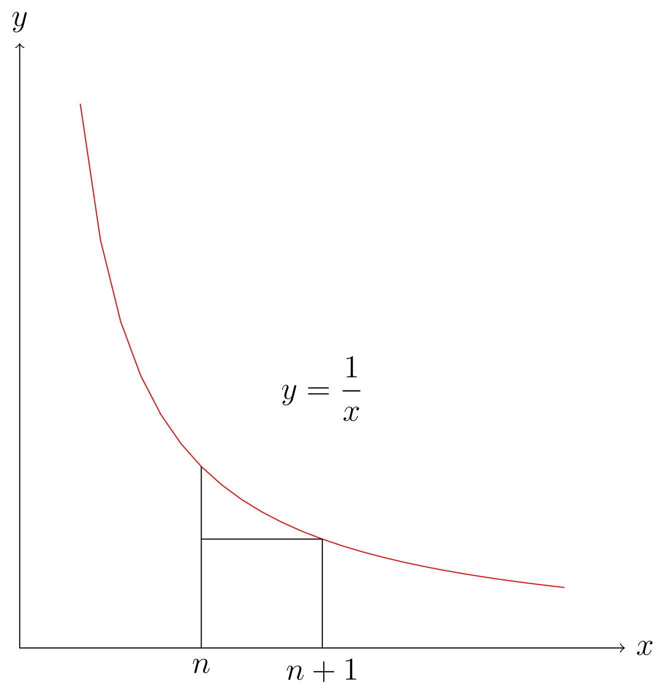


Figure 2: Comparing one box with $\ln x$.

We now claim that the sequence decreases. For this observe that

$$a_n - a_{n+1} = (\ln(n+1) - \ln n) - \frac{1}{n+1}.$$

From Figure 2, we see that the area under the curve $y = 1/x$, from $x = n$ to $x = n+1$, is larger than the area of the box, which is $1/(n+1)$. This means that

$$a_n - a_{n+1} = (\ln(n+1) - \ln n) - \frac{1}{n+1} > 0,$$

hence a_n decreases.

Since every decreasing sequence which is bounded from below converges, the limit of a_n as n goes to infinity exists. This limit value has a special name. It is called the Euler constant, or the Euler-Mascheroni constant. It is customary to denote this limit by γ.

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = \gamma = 0.5772156649...$$

It is not known yet whether γ is a rational number or not!

ii. Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. From the first part, we know that

$$\lim_{n\to\infty} (H_n - \ln n) = \gamma, \quad \text{or equivalently} \quad \lim_{n\to\infty} \left(H_{n^2} - \ln n^2\right) = \gamma.$$

Now we observe that

$$\begin{aligned} a_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{n^2} \\ &= H_n - (H_{n^2} - H_n) \\ &= 2H_n - H_{n^2} \\ &= 2(H_n - \ln n) - (H_{n^2} - \ln n^2) \\ &\to 2\gamma - \gamma = \gamma \quad \text{as} \quad n \to \infty. \end{aligned}$$

This new property of γ is due to J. J. Mačys, The American Mathematical Monthly, Volume 119, No: 1, page 82, year 2012.

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Q-2) Let $\alpha \geq 0$ be any non-negative real number. Define a sequence a_n recursively as follows.

$$a_1 = \alpha, \; a_{n+1} = \sqrt{1 + 2a_n} \quad \text{for} \quad n \geq 1.$$

For which values of α does the sequence a_n converge? Find $\lim_{n \to \infty} a_n$ when it exists.

Solution:

First assume that $\lim_{n \to \infty} a_n$ exists. Denote the limit by L. Then by taking the limit of both sides of the recursive relation $a_{n+1} = \sqrt{1 + 2a_n}$ as n goes to infinity, we find that

$$L = \sqrt{1 + 2L}, \quad \text{or} \quad L = 1 + \sqrt{2}.$$

This shows that if the limit exists, then it is $1 + \sqrt{2}$. But does the limit exist?

We check if the sequence is increasing or decreasing. The sign of $a_{n+1} - a_n$ is the same as the sign of $a_{n+1}^2 - a_n^2$. Using the recursive relation, we find that

$$a_{n+1}^2 - a_n^2 = 2(a_n - a_{n-1}), \quad \text{for} \quad n > 1.$$

By induction, the sign of $a_{n+1} - a_n$ is the same as the sign of $a_2 - a_1$, which in turn is the same as the sign of $a_2^2 - a_1^2$. We find that

$$a_2^2 - a_1^2 = 1 + 2\alpha - \alpha^2 = \begin{cases} > 0 & \text{for } 0 \leq \alpha < 1 + \sqrt{2}, \\ = 0 & \text{for } \alpha = 1 + \sqrt{2}, \\ < 0 & \text{for } \alpha > 1 + \sqrt{2}. \end{cases}$$

We examine these cases separately.

Case 1: $0 \leq \alpha < 1 + \sqrt{2}$.
In this case, as we showed above, the sequence a_n is increasing. We claim that each $a_n < 1 + \sqrt{2}$. Clearly $a_1 = \alpha < 1 + \sqrt{2}$. Assume $a_n < 1 + \sqrt{2}$. Then

$$a_{n+1}^2 = 1 + 2a_n < 1 + 2(1 + \sqrt{2}) = (1 + \sqrt{2})^2.$$

Thus $a_{n+1} < 1 + \sqrt{2}$, and this proves our claim.

We now know that a_n is an increasing sequence which is bounded from above, so the limit exists.

Case 2: $\alpha = 1 + \sqrt{2}$.
In this case $a_n = 1 + \sqrt{2}$ for all $n \geq 1$, so the limit exists.

Case 3: $\alpha > 1 + \sqrt{2}$.
In this case the sequence a_n is decreasing. We claim that each $a_n > 1+\sqrt{2}$. Clearly $a_1 = \alpha > 1+\sqrt{2}$. Assume $a_n > 1 + \sqrt{2}$. Then

$$a_{n+1}^2 = 1 + 2a_n > 1 + 2(1 + \sqrt{2}) = (1 + \sqrt{2})^2.$$

Thus $a_{n+1} > 1 + \sqrt{2}$, and this proves our claim.

We now know that a_n is an decreasing sequence which is bounded from below, so the limit exists.

We proved that for all $\alpha \geq 0$, the sequence a_n converges to $1 + \sqrt{2}$.

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Q-3)

i. For which values of $\alpha \in \mathbb{R}$ does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\alpha}}$ converge?

ii. For which values of $\alpha \in \mathbb{R}$ does the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln \ln n)^{\alpha}}$ converge?

Solution:

i. We observe that

$$\lim_{n\to\infty} \frac{\dfrac{1}{(\ln n)^{\alpha}}}{\dfrac{1}{n}} = \infty, \quad \text{for all} \quad \alpha \in \mathbb{R}.$$

By limit comparison test, we conclude that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\alpha}}$ diverges for all $\alpha \in \mathbb{R}$.

ii. We use integral test for this series. We have

$$\int_{3}^{\infty} \frac{dx}{x(\ln \ln x)^{\alpha}} = \int_{\ln 3}^{\infty} \frac{du}{(\ln u)^{\alpha}}.$$

But this last integral diverges by integral test where we compare it by the series in part (i). Hence $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln \ln n)^{\alpha}}$ diverges for all α.

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Q-4)

i. Find $\lim_{n\to\infty} \frac{n!}{n^n}$. Let $a_n = \frac{n!}{n^n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge?

ii. Does the series $\sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$ converge?

Solution:

i. Since $\frac{n!}{n^n} = \frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n} < \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ by the sandwich theorem.

To check the convergence of the infinite sum, we use the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1,$$

so the series converges.

ii. Let $a_n = \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$. Then observe that

$$\frac{1}{a_n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + 1 + \cdots + 1 = n,$$

so that $a_n > \frac{1}{n}$, and the series $\sum_{n=1}^{\infty} a_n$ diverges by comparison with the harmonic series.