Due Date: March 30, 2012 Friday class time NAME:..

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Math 114 Calculus – Homework 3 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are 4 questions on your booklet. Write your name on top of every page.
Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.
Everything you write on your paper should be part of a well constructed sentence. No hanging equations will be read. No sequence of equations will be read unless they are part of a well constructed, meaningful sentence.

Q-1) Let $f(x,y)$ be a function defined in some open neighborhood of (x_0,y_0). Assume that there exist constants A and B such that f satisfies one of the following conditions DIFF1 or DIFF2.

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0,y_0)-Ah-Bk}{\sqrt{h^2+k^2}} = 0. \qquad \text{(DIFF1)}$$

$$f(x_0+h,y_0+k) = f(x_0,y_0)+Ah+Bk+\epsilon_1 h+\epsilon_2 k, \qquad \text{(DIFF2)}$$

where ϵ_i is a function of h and k such that $\lim_{(h,k)\to(0,0)} \epsilon_i = 0$, $i=1,2$.

(i) Show that if f satisfies the condition DIFF1, then $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist and $A = f_x(x_0,y_0)$, $B = f_y(x_0,y_0)$.

(ii) Show that if f satisfies the condition DIFF2, then $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist and $A = f_x(x_0,y_0)$, $B = f_y(x_0,y_0)$.

(iii) Show that the conditions DIFF1 and DIFF2 are equivalent.

Remark: A function satisfying any of the equivalent conditions DIFF1 or DIFF2 is called differentiable at (x_0,y_0).

Solution:

(i)

$$\begin{aligned}
f_x(x_0,y_0) &= \lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \\
&= \lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)-Ah+Ah}{h} \\
&= \lim_{h\to0}\left(\frac{f(x_0+h,y_0)-f(x_0,y_0)-Ah}{h}+A\right) \\
&= \lim_{\substack{(h,k)\to(0,0)\\k=0}}\left(\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-Ah-Bk}{\sqrt{h^2+k^2}}+A\right) \\
&= A.
\end{aligned}$$

Similarly $f_y(x_0,y_0)=B$.

(ii)

$$\begin{aligned}
f_x(x_0,y_0) &= \lim_{h\to0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \\
&= \lim_{h\to0}\frac{Ah+\epsilon_1(h,0)h}{h} \\
&= A+\lim_{h\to0}\epsilon_1(h,0) \\
&= A.
\end{aligned}$$

Similarly $f_y(x_0,y_0)=B$.

(iii)

(DIFF1$\Rightarrow$ DIFF2)
Define a new function

$$\epsilon(h,k) = \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - Ah - Bk}{\sqrt{h^2+k^2}} \text{ when } (h,k) \neq (0,0), \text{ and } \epsilon(0,0) = 0,$$

where A and B are as in condition DIFF1. Also define a function of h and k simply as

$$s = \begin{cases} +1 & \text{if } hk \geq 0, \\ -1 & \text{if } hk < 0. \end{cases}$$

Finally consider the auxiliary function α defined as

$$\alpha(t) = \sqrt{1+t^2} - t = \frac{1}{\sqrt{1+t^2}+t}, \text{ for } t \geq 0.$$

Clearly $\alpha(t) > 0$ and $\lim_{t\to\infty} \alpha(t) = 0$. This shows that $\alpha(t)$ is bounded; there exists a number M such that

$$0 < \alpha(t) \leq M, \text{ for } t \geq 0.$$

We now proceed to show that the condition DIFF2 holds. We claim that the ϵ_1 and ϵ_2 of DIFF2 are the following functions.

$$\begin{aligned} \epsilon_1(h,k) &= \epsilon(h,k)s, \\ \epsilon_2(h,k) &= \epsilon(h,k)\alpha(|h/k|), \text{ if } k \neq 0, \\ \epsilon_2(h,0) &= 0. \end{aligned}$$

Clearly it follows from DIFF1 that

$$\lim_{(h,k)\to(0,0)} \epsilon_1(h,k) = 0.$$

We also have, when $k \neq 0$,

$$0 \leq |\epsilon_2(h,k)| = |\epsilon(h,k)||\alpha(|h/k|)| \leq M|\epsilon(h,k)|.$$

It follows form the condition DIFF1 and the sandwich theorem that

$$\lim_{(h,k)\to(0,0)} \epsilon_2(h,k) = 0.$$

Observe that

$$\epsilon(h,k)\sqrt{h^2+k^2} = \epsilon_1(h,k)h + \epsilon_2(h,k)k,$$

which is precisely the condition DIFF2.

(DIFF2⇒ DIFF1)

$$\begin{aligned}\left|\frac{f(x_0+h, y_0+k) - f(x_0, y_0) - Ah - Bk}{\sqrt{h^2+k^2}}\right| &= \left|\frac{\epsilon_1(h,k)h + \epsilon_2(h,k)k}{\sqrt{h^2+k^2}}\right| \\ &\leq |\epsilon_1(h,k)|\frac{|h|}{\sqrt{h^2+k^2}} + |\epsilon_2(h,k)|\frac{|k|}{\sqrt{h^2+k^2}} \\ &\leq |\epsilon_1(h,k)| + |\epsilon_2(h,k)|,\end{aligned}$$

which goes to zero as $(h, k) \to (0, 0)$, giving us the condition DIFF1.

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Q-2) Define

$$f(x,y) = \begin{cases} \dfrac{y^5 - x^2 y}{x^2 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(i) Show that f is continuous at $(0,0)$.

(ii) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.

(iii) Show that f is not differentiable at $(0,0)$.

Solution:

(i)
$\dfrac{y^5}{x^2 + y^4}$ is continuous at the origin since $\dfrac{0}{2} + \dfrac{5}{4} > 1$.
$\dfrac{x^2 y}{x^2 + y^4}$ is continuous at the origin since $\dfrac{2}{2} + \dfrac{1}{4} > 1$.
Hence f, being the difference of two continuous functions, is continuous.

(ii)

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{0 - 0)}{x} = 0,$$
$$f_y(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{x \to 0} \frac{y - 0)}{y} = 1.$$

(iii)
Let

$$\phi(x,y) = \frac{f(x,y) - f(0,0) - 0 \cdot x - 1 \cdot y}{\sqrt{x^2 + y^2}} = -2 \cdot \frac{x^2 y}{(x^2 + y^2)^{3/2}}.$$

Then

$$\lim_{x \to 0} \phi(x, \lambda x) = \frac{-2\lambda}{\sqrt{1 + \lambda^2}}.$$

Since this limit depends on path, the general limit does not exist and the function f is not differentiable at the origin.

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Q-3) Define

$$g(x,y) = \begin{cases} \dfrac{y^5 + x^2 y}{x^2 + y^4} & \text{if } (x,y) \neq 0, \\ 0 & \text{if } (x,y) = 0. \end{cases}$$

(i) Show that g is continuous at $(0,0)$.

(ii) Show that $g_x(0,0)$ and $g_y(0,0)$ exist.

(iii) Show that g is differentiable at $(0,0)$.

Solution:

Here the function is $g(x,y) = y$ and the problem is trivial.

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Q-4) Define

$$h(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq 0, \\ 0 & \text{if } (x,y) = 0. \end{cases}$$

(i) Find h_x, h_y, h_{xy}, h_{yx} at points $(x,y) \neq (0,0)$.

(ii) Find h_x, h_y, h_{xy}, h_{yx} at points $(x,y) = (0,0)$.

(iii) Did you get $h_{xy}(0,0) = h_{yx}(0,0)$? Explain why?.

Solution:

(i)

$$h_x = \frac{y\left(x^4 + 4\,x^2y^2 - y^4\right)}{\left(x^2+y^2\right)^2}, \qquad h_y = \frac{x\left(x^4 - 4\,x^2y^2 - y^4\right)}{\left(x^2+y^2\right)^2}$$

$$h_{xy} = \frac{x^6 + 9\,x^4y^2 - 9\,x^2y^4 - y^6}{\left(x^2+y^2\right)^3} = h_{yx}$$

(ii)

$$h_x(0,0) = \lim_{x\to 0} \frac{h(x,0) - h(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0.$$

Similarly $h_y(0,0) = 0$.

$$h_{xy}(0,0) = \lim_{y\to 0} \frac{h_x(0,y) - h_x(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1.$$

$$h_{yx}(0,0) = \lim_{x\to 0} \frac{h_y(0,y) - h_y(0,0)}{y} = \lim_{x\to 0} \frac{x-0}{x} = 1.$$

(iii)
We did not get $h_{xy}(0,0) = h_{yx}(0,0)$. This is not surprising as $h_{xy}(x,y)$ is not continuous at the origin. This can be seen by observing that

$$h_{xy}(x, \lambda x) = \frac{1 + 9\lambda^2 - 9\lambda^4 - \lambda^6}{(1+\lambda^2)^3}.$$

Hence the limit along different lines give different limits at the origin, from which we conclude that h_{xy} is not continuous at the origin.