Date: 14 March 2012, Wednesday
Time: 08:40-10:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 114 Calculus – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.
Every mathematical symbol and every equation you write must be part of a well constructed sentence. I will not read any hanging equations or symbols. I will not try to interpret your symbols. I don't do mind reading, yet.

Use responsibly:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^n\frac{1}{(2n+1)!}x^{2n+1} + \cdots$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n\frac{1}{(2n)!}x^{2n} + \cdots$$
$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + (-1)^{n+1}\frac{1}{2n+1}x^{2n+1} + \cdots$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

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Q-1)

(i) Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be infinite series with positive terms. Assume further that $\sum_{n=0}^{\infty} b_n$ converges and that

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},$$

for all $n \geq N$ for some fixed N. Show that $\sum_{n=0}^{\infty} a_n$ also converges.

(ii) Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with positive terms. Assume that there exists a real number $p > 1$ such that

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n},$$

for all $n \geq N$ for some fixed N. Show that $\sum_{n=0}^{\infty} a_n$ converges.

Hint: You may find Bernoulli's inequality useful: For all $p > 1$ and $0 < x < 1$, we have $1 - px \leq (1-x)^p$. You may also need part (i).

Solution:

(i) It follows that for all $n \geq N$, we have

$$\frac{a_n}{b_n} \leq \frac{a_N}{b_N} = K,$$

which implies

$$0 \leq a_n \leq K b_n,$$

for all large n. Hence $\sum a_n$ converges by direct comparison.

(ii) We have for all large n,

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n} \leq (1 - \frac{1}{n})^p = \frac{1/n^p}{1/(n-1)^p}.$$

Since $\sum 1/(n-1)^p$ converges when $p > 1$, our series $\sum a_n$ converges by the result of part (i).

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Q-2) Let $f(x) = x^2 \ln x$, and let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

be the Taylor series of f at $x = 1$. Write explicitly the value of a_n for $n = 0, 1, \ldots,$ and find the interval on which the above Taylor series converges to the function f.

Solution:

First observe that

$$f'(x) = 2x \ln x + x,\ \ f''(x) = 2\ln x + 3,\ \ f^{(n)}(x) = \frac{(-1)^{n+1} 2 \cdot (n-3)!}{x^{n-2}}, n \geq 3.$$

It follows that

$$f'(1) = 0, f''(1) = 1, f^{(n)}(1) = (-1)^{n+1} 2 \cdot (n-3)!, n \geq 3,$$

and hence

$$a_0 = 0, a_1 = 1, a_2 = \frac{3}{2}, a_n = \frac{(-1)^{n+1} 2}{n(n-1)(n-2)}, n \geq 3.$$

To find the radius of convergence of the series we try ratio test.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}(x-1)^{n+1}}{a_n (x-1)^n} \right| = |x-1|,$$

which says that the series converges for $0 < x < 2$. When $x = 2$, the series becomes an alternating series so converges and hence the interval of convergence of the series is $0 < x \leq 2$. However, the Taylor error term is easily estimated to go to zero as n goes to infinity only when $1/2 \leq x \leq 2$. The usual estimates of the error does not necessarily imply that it goes to zero when $0 < x < 1/2$. Therefore we need to develop the series using an alternate approach to keep track of the Taylor error term.

For this we can start with

$$\ln(1+t) = t - \frac{t^2}{2} + \cdots + (-1)^{n+1} \frac{t^n}{n} + \cdots, -1 < t \leq 1,$$

and write

$$x^2 \ln x = [(x-1)+1]^2 \ln[1 + (x-1)]$$

to use the series expansion of $\ln(1+t)$. This immediately gives the convergence of the series to $f(x)$ for $0 < x \leq 2$.

Here is the series expansion.

$$x^2 \ln x = (x-1) + \frac{3}{2}(x-1)^2 + \sum_{n=3}^{\infty} (-1)^{n+1} \frac{2}{n(n-1)(n-2)} (x-1)^n, 0 < x \leq 2.$$

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Q-3) Find $\lim_{n\to\infty} \frac{n!2^n}{n^n}$.

Solution:

Let $a_n = \frac{n!2^n}{n^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2}{(1+1/n)^n} \to \frac{2}{e} < 1 \text{ as } n \to \infty.$$

Therefore the series $\sum_{n=1}^{\infty} a_n$ converges and necessarily we must have $\lim_{n\to\infty} \frac{n!2^n}{n^n} = 0$.

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Q-4) Let $\tan(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$ be the Taylor expansion of $\tan x$ at $x = 0$. Find $a_0, \ldots, a_6$.
Also find the interval where the above Taylor series converges to $\tan x$.

Solution:

Since $\tan x$ is an odd function, we immediately know that $a_0 = a_2 = a_4 = a_6 = 0$. To calculate the remaining Taylor coefficients using derivatives is cumbersome. However we can use the trivial observation that

$$\tan x \cos x = \sin x,$$

and substitute here the corresponding power series to find

$$(a_1x + a_3x^3 + a_5x^5 + \cdots)(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots .$$

This gives

$$a_1x + \left(-\frac{a_1}{2} + a_3\right)x^3 + \left(\frac{a_1}{24} - \frac{a_3}{2} + a_5\right)x^5 + \cdots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots ,$$

from where we find $a_1 = 1, a_3 = \dfrac{1}{3}, a_5 = \dfrac{2}{15}$.

The Taylor series converges around $x = 0$ up to the first singularity. Since cosine vanishes at $x = \pi/2$, the interval of convergence of the Taylor series of $\tan x$ around $x = 0$ is $(-\pi/2, \pi/2)$.

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Q-5) Find $\lim_{x\to 0} \dfrac{(\arctan x - x)(\sinh x - x)(1 - x^2 \ln x)}{(3\tan x - 3x - x^3)(\sec^3 x - x^3)(e^x - 1)}$.

Solution:

We first notice that $\lim_{x\to 0} \dfrac{(1 - x^2 \ln x)}{(\sec^3 x - x^3)} = 1$ and does not contribute to the indeterminacy of the above limit. So it suffices to examine the remaining factors. For this we see that both the numerator and the denominator vanish to the order of 6. Then we write

$$
\begin{aligned}
&\lim_{x\to 0} \frac{(\arctan x - x)(\sinh x - x)}{(3\tan x - 3x - x^3)(e^x - 1)} \\
&= \lim_{x\to 0} \frac{(-\frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots)(\frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots)}{(\frac{2}{5}x^5 + \frac{17}{105}x^7 + \cdots)(x + \frac{1}{2}x^2 + \cdots)} \\
&= \lim_{x\to 0} \frac{-\frac{1}{18}x^6 + \frac{11}{360}x^8 + \cdots}{\frac{2}{5}x^6 + \frac{1}{5}x^7 + \cdots} \\
&= -\frac{5}{36},
\end{aligned}
$$

which is the required limit.