Due Date: November 18, 2008, Tuesday NAME:..

Math 123 Abstract Mathematics I
Homework 1 – Solutions

In the following problems assume only the validity of Peano axioms **P1-P7** as stated on pages 106 and 110. The hint of problem 1 can be used, after slight modification, also for the other problems.

1) Show that addition is associative.
Hint: For $n \in \mathbb{N}$ let $P(n)$ be the statement that $\forall\ m, k \in \mathbb{N},\ (m+k)+n = m+(k+n)$. Then prove $P(n)$ by induction for all $n \in \mathbb{N}$.

2) Show that for all $m \in \mathbb{N},\ 1+m = m+1$.

3) Show that addition in $\mathbb{N}$ is commutative.

4) Show that cancellation holds for addition in $\mathbb{N}$, i.e. for all $k, m, n \in \mathbb{N},\ m+n = k+n \implies m = k$.

Solutions:
1) We use the hint. First we show that $P(1)$ is true. ($\overset{P6}{=}$ means that Peano axiom no 6 is used for the equality, and $\overset{P(n)}{=}$ means that the induction hypothesis $P(n)$ is used.)
$(m+k)+1 \overset{P6}{=} (m+k)' \overset{P7}{=} m+k' \overset{P6}{=} m+(k+1)$. This establishes the validity of $P(1)$.

Next we assume $P(n)$, i.e. $(m+k)+n = m+(k+n)$.

Next we check $P(n')$.

$(m+k)+n' \overset{P7}{=} ((m+k)+n)' \overset{P(n)}{=} (m+(k+n))' \overset{P7}{=} m+(k+n)' \overset{P7}{=} m+(k+n')$. This establishes the validity of $P(n')$, and this completes the proof of associativity.

2) Let $P(m)$ be the statement that $1+m = m+1$.

When $m = 1$ we have the trivial relation $1+1 = 1+1$, so $P(1)$ holds.

Assume $P(n)$ and check $P(n')$. (Here $\overset{Q1}{=}$ means the result of question 1 above is used.)

$1+m' \overset{P6}{=} 1+(m+1) \overset{associativity}{=} (1+m)+1 \overset{P(1)}{=} (m+1)+1 \overset{P6}{=} m'+1$. Thus $P(n')$ holds and the proof is complete.

3) Let $P(n)$ be the statement that for all natural numbers m we have $m+n = n+m$.

$P(1)$ is established in problem 1.

Assume $P(n)$ and check for $P(n')$. (Here $\overset{Q2}{=}$ means the result of question 2 above is used.)

$m+n' \overset{P6}{=} m+(n+1) \overset{Q1}{=} (m+n)+1 \overset{P6}{=} (m+n)' \overset{Q2}{=} (n+m)' \overset{P7}{=} n+m' \overset{P6}{=} n+(m+1) \overset{Q2}{=} n+(1+m) \overset{Q1}{=} (n+1)+m \overset{P6}{=} n'+m$. This establishes $P(n')$ and completes the proof.

4) Let $P(n)$ be the statement "If m and k are natural numbers and if $m + n = k + n$ for a natural number n, then $m = k$".

$P(1)$ is exactly $P4$, the Peano axiom 4.

Assume $P(n)$ and check for $P(n')$.

If $m+n' = k+n'$, then by $P7$ $(m+n)' = (k+n)'$. From this by $P4$ we get $m+n = k+n$ and this implies by $P(n)$ that $m = k$, which establishes $P(n')$ and completes the proof.

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