Math 206 Complex Calculus
Quiz-2
<u>Solutions</u>

March 7, 2002 Ali Sinan Sertöz

1) Find all values of z^c and show which one is the principal value, where z and c are given as:
(a) $z = 2e^{i5\pi/4}$, $c = 1 + i$. **(b)** $z = -1 - \sqrt{3}i$, $c = \sqrt{3}i$.

Solution: In general $z^c = \exp(c\ \log z)$ and the principal value is obtained when you take the argument of z to lie between $-\pi$ and π.
Solution a:

$$
\begin{aligned}
z^c &= \exp((1+i)\log(2e^{i(5\pi/4+2n\pi)}), \quad n \in \mathbb{Z} \\
&= \exp((1+i)(\ln 2 + i(5\pi/4 + 2n\pi))) \\
&= \exp[(\ln 2 - (\frac{5\pi}{4} + 2n\pi)) + i(\ln 2 + \frac{5\pi}{4} + 2n\pi)] \\
&= \exp[\ln 2 - (\frac{5\pi}{4} + 2n\pi)][\cos(\ln 2 + \frac{5\pi}{4} + 2n\pi) + i\sin(\ln 2 + \frac{5\pi}{4} + 2n\pi)]
\end{aligned}
$$

Principal value is obtained when n is such that $-\pi < \dfrac{5\pi}{4} + 2n\pi < \pi$. This is satisfied for $n = -1$ and then the principal argument of z becomes $-\dfrac{3\pi}{4}$.
Hence the principal value for z^c is
$\exp[\ln 2 + \dfrac{3\pi}{4}][\cos(\ln 2 - \dfrac{3\pi}{4}) + i\sin(\ln 2 - \dfrac{3\pi}{4})] \cong -1.9 - 21i$.

Solution b: $z = -1 - \sqrt{3}i = 2e^{(-2\pi/3+2n\pi)}$, $n \in \mathbb{Z}$.

$$
\begin{aligned}
z^c &= \exp(\sqrt{3}i\log 2e^{(-2\pi/3+2n\pi)}) \\
&= \exp(\sqrt{3}i(\ln 2 + i(-2\pi/3 + 2n\pi))) \\
&= \exp(-\sqrt{3}(-2\pi/3 + 2n\pi) + i\sqrt{3}\ln 2) \\
&= \exp(-\sqrt{3}(-2\pi/3 + 2n\pi))[\cos(\sqrt{3}\ln 2) + i\sin(\sqrt{3}\ln 2)]
\end{aligned}
$$

The principal value here corresponds to $n = 0$:
$z^c = \exp(2\sqrt{3}\pi/3)[\cos(\sqrt{3}\ln 2) + i\sin(\sqrt{3}\ln 2)] \cong 13.6 + 35i$.

(The numerical calculation was not required in the quiz.)