Math 206 Complex Calculus
Quiz-3
Solutions

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1-a) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}dx$.

Solution 1-a: Let $f(z) = \frac{z^2}{1+z^4}$. Let C_R denote the closed contour which consists of the path along the real axis from $-R$ to R, followed by the semi-circle $|z| = R$, where $R > 1$. The poles of the function f are the points where $1 + z^4 = 0$, and there are two of them inside the contour C_R. They are

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \\ z_2 &= \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}. \end{aligned}$$

Let $g(z) = \dfrac{x^2}{4x^3} = \dfrac{1}{4x}$. Then the residue of f at z_k is $g(z_k)$, $k = 1, 2$.

$$\begin{aligned} g(z_1) &= \frac{1}{4}(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) \\ g(z_2) &= \frac{1}{4}(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) \\ g(z_1) + g(z_2) &= -\frac{i\sqrt{2}}{4}. \end{aligned}$$

Since $\deg z^2 < (\deg(1 + z^4)) - 1$, the integral on the semicircle goes to zero as R goes to infinity. The integral along $[-R, R]$ converges to the Cauchy Principal Value which is twice the integral we are trying to evaluate, since the integrant is an even function. Thus we get

$$\begin{aligned} \int_0^\infty \frac{x^2}{1+x^4}dx &= \left(2\pi i[g(z_1) + g(z_2)]\right)/2 \\ &= \frac{\pi\sqrt{2}}{4}. \end{aligned}$$

1-b) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^4}{1+x^6}dx$.

Solution 1-b: Let $f(z) = \frac{z^4}{1+z^6}$. Let C_R denote the closed contour which consists of the path along the real axis from $-R$ to R, followed by the semicircle $|z| = R$, where $R > 1$. The poles of the function f are the points where $1 + z^6 = 0$, and there are three of them inside the contour C_R. They are

$$\begin{aligned} z_1 &= \frac{\sqrt{3}}{2} + \frac{i}{2}, \\ z_2 &= i, \\ z_3 &= \frac{-\sqrt{3}}{2} + \frac{i}{2}. \end{aligned}$$

Let $g(z) = \dfrac{x^4}{6x^5} = \dfrac{1}{6x}$. Then the residue of f at z_k is $g(z_k)$, $k = 1, 2, 3$.

$$\begin{aligned} g(z_1) &= \frac{1}{6}(\frac{\sqrt{3}}{2} - \frac{i}{2}), \\ g(z_2) &= -i/6, \\ g(z_3) &= \frac{1}{6}(-\frac{\sqrt{3}}{2} - \frac{i}{2}), \\ g(z_1) + g(z_2) + g(z_3) &= -\frac{i}{3}. \end{aligned}$$

Since $\deg z^4 < (\deg(1 + z^6)) - 1$, the integral on the semicircle goes to zero as R goes to infinity. The integral along $[-R, R]$ converges to the Cauchy Principal Value which is twice the integral we are trying to evaluate, since the integrant is an even function. Thus we get

$$\begin{aligned} \int_0^\infty \frac{x^4}{1+x^6}dx &= \left(2\pi i[g(z_1) + g(z_2) + g(z_3)]\right)/2 \\ &= \frac{\pi}{3}. \end{aligned}$$