Date: 26 March 2004, Friday

Math 206 Complex Calculus – Homework5 - Solutions

p149 ex 6

When $f(z) = \cos z$, $f^{(n)}(z) = \dfrac{d^n}{dz^n}(\cos z) = \begin{cases} \cos z & \text{if } n \equiv 0 \bmod 4 \\ -\sin z & \text{if } n \equiv 1 \bmod 4 \\ -\cos z & \text{if } n \equiv 2 \bmod 4 \\ \sin z & \text{if } n \equiv 3 \bmod 4 \end{cases}$

Therefore $f^{(2n)}(\pi/2) = 0$ and $f^{(2n+1)}(\pi/2) = (-1)^{n+1}$. The Taylor series of $\cos z$ around $\pi/2$ is then

$$\cos z = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}(z - \pi/2)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(z - \pi/2)^{2n+1}.$$

p149 ex 8

The Maclaurin series of $\tanh z = \frac{\sinh z}{\cosh z}$ converges for all values of z for which $|z|$ is less than the modulus of that zero of $\cosh z$ which is closest to the origin. We know, from page 74 formula (15), that the nearest zero has modulus $\pi/2$.

Therefore $\tanh z = z - \dfrac{1}{3}z^3 + \dfrac{2}{15}z^5 - \dfrac{17}{315}z^7 + \dfrac{62}{2835}z^9 + \cdots$ for $|z| < \pi/2$.

p149 ex 10

From equation (1) of page 146 we have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots.$$

Dividing both sides by z^2 we have

$$\frac{e^z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots.$$

The other parts of this exercise are done similarly. The required expansions of the relevant functions, $\sin z$, $\sinh z$ and $\cosh z$ are on page 147, equations (2), (3) and (5) respectively.

p157 ex 8

$$\begin{aligned} \frac{a}{z-a} &= \frac{a}{z}\frac{1}{1-\frac{a}{z}} \\ &= \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n \qquad \text{since} \quad |a/z| < 1, \\ &= \sum_{n=1}^{\infty}\frac{a^n}{z^n}. \end{aligned}$$

For the second part, put $z = e^{i\theta}$ on both sides

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n}$$

$$\begin{aligned}
\frac{a}{e^{i\theta}-a} &= \sum_{n=1}^{\infty} \frac{a^n}{e^{in\theta}} \\
\frac{a}{\cos\theta + i\sin\theta - a} &= \sum_{n=1}^{\infty} a^n e^{-in\theta} \\
\frac{a\cos\theta - a^2}{1-2a\cos\theta + a^2} - i\frac{a\sin\theta}{1-2a\cos\theta+a^2} &= \sum_{n=1}^{\infty} a^n \cos n\theta - i\sum_{n=1}^{\infty} a^n \sin n\theta.
\end{aligned}$$

p158 ex 11
Let us first rewrite equations (1), (2) and (3) of page 150 as follows, keeping in mind that for this problem $z_0 = 0$:

(1) $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$, where

(2) $a_n = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w^{n+1}}$, $(n = 0, 1, 2, \ldots)$ and

(3) $b_n = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w^{-n+1}}$, $(n = 1, 2, 3\ldots)$.

Here w is on the unit circle so $w = e^{i\phi}$ with $-\pi \le \phi \le \pi$ and z is the usual parameter with $z = e^{i\theta}$. Beware that we have to use different notation from the book to avoid confusing the parameter of integration with the parameter of the function. Using these parameters we put equations (2) and (3) into (1) to get

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w^{n+1}}\right] z^n + \sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w^{-n+1}}\right]\frac{1}{z^n} \\
&= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\left(\frac{1}{e^{i\phi}}\right)^n d\phi\right] z^n + \sum_{n=1}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\left(e^{i\phi}\right)^n d\phi\right]\frac{1}{z^n} \\
&= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\left(\frac{z}{e^{i\phi}}\right)^n d\phi\right] + \sum_{n=1}^{\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\left(\frac{e^{i\phi}}{z}\right)^n d\phi\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi.
\end{aligned}$$

For the second part observe that

$$\begin{aligned}
\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n &= \left(\frac{e^{i\theta}}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{e^{i\theta}}\right)^n \\
&= e^{in(\theta-\phi)} + e^{in(\phi-\theta)} \\
&= 2\cos[n(\theta-\phi)].
\end{aligned}$$

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