Date: May 24, 2005; Tuesday NAME:..
Instructors: Sertöz and Özgüler
Time: 16.00-18.00 STUDENT NO:..

Math 206 Complex Calculus–Final Exam

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet.
Write your name on the top of every page.

Q-1) Determine the inverse Z-transform of

$$F(z) = \frac{e^{1/z}}{z-2}.$$

Solution: Use the method of residues: $f(n) = \sum Res[z^{n-1}F(z)]$. Two singularitites of $z^{n-1}F(z)$ are at $z=0$ and at $z=2$, both simple. Now, $Res_{z=0}[z^{n-1}F(z)]$ is the coefficient of z^{-1} in the product of series

$$z^{-1}\frac{1}{1-(2/z)} = z^{-1} + 2z^{-2} + 2^2z^{-3} + ... + 2^{n-1}z^{-n} + 2^n z^{-(n+1)} + 2^{n+1}z^{-(n+2)} + ...$$

and

$$z^{n-1}e^{1/z} = z^{n-1} + z^{n-2} + \frac{1}{2!}z^{n-3} + ... + \frac{1}{(n-1)!} + \frac{1}{n!}z^{-1} + \frac{1}{(n+1)!} + z^{-2} + ...$$

which is

$$\sum_{k=0}^{k=n-1} \frac{2^k}{(n-k-1)!}.$$

On the other hand, $Res_{z=2}[z^{n-1}F(z)] = 2^{n-1}e^{0.5}$ so that

$$f(n) = \sqrt{e}2^{n-1} + \sum_{k=0}^{k=n-1} \frac{2^k}{(n-k-1)!}.$$

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Q-2) Consider the sequence $1, 1, 2, 4, 7, 11, 16, ...$ that begins with $n = 0$ and satisfies $f(n+1) - f(n) = n$. Find $f(n)$.

Solution: $zF(z) - z + F(z) = z/(z-1)^2$ gives $F(z) = z/(z-1)^3 + z/(z-1)$. Now, $Z^{-1}\{z/(z-1)^3\} = Res_{z=1}[z^n/(z-1)^3] = n(n-1)/2$. Also, $Z^{-1}\{z/(z-1)\} = 1$. Hence,

$$f(n) = \frac{n(n-1)}{2} + 1.$$

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Q-3) Find a conformal map which maps the interior of the set $\{x+iy \in \mathbb{C} | y \geq 0, 0 \leq x \leq \pi/2\}$ onto the interior of the unit disk such that the point $\pi/4 + i$ is mapped to the origin.

Solution: $w_1 = \sin z$ maps the region onto the first quadrant. $w_2 = w_1^2$ maps the first quadrant onto the upper half plane. $w = (w_2 - z_0)/(w_2 - \overline{z_0})$ maps the first quadrant onto the unit circle such that z_0 is mapped to the origin. We don't need the $\exp(i\alpha)$ factor. Now follow what happens to the given point to find precisely what z_0 should be. It turns out that $z_0 = \frac{1}{2}(1 + i \sinh 2)$.

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Q-4) Using a complex logarithmic mapping,
a) find a bounded harmonic function $H(x,y)$, or $H(r,\theta)$, in the wedge $0 < arg(z) < \pi/6$, $|z| > 0$ such that $H(r,0) = 0$ and $H(r,\pi/6) = 1$ for $r > 0$.
b) Find a harmonic conjugate $G(x,y)$ of $H(x,y)$ and describe the families of level curves $H(x,y) = c_1$, $G(x,y) = c_2$ for real constants c_1, c_2.

Solution: a) $H(u,v) = Re\{-i\frac{6}{\pi}Log\, z\}$ in w-plane so that $H(x,y) = \frac{6}{\pi}arctan(y/x)$ with the range of arctan function taken between 0 and π.
b) $G(u,v) = Im\{-i\frac{6}{\pi}Log\, z\}$ in w-plane so that $G(x,y) = \frac{6}{\pi}ln(\sqrt{x^2+y^2})$. Hence, $H(x,y) = c_1$ give radial lines and $G(x,y) = c_2$ give circular arcs in the wedge.