Math 206 – Homework #1
Solutions

Q1) $\left.\begin{aligned} z_1 &= x_1 + i y_1 \\ z_2 &= x_2 + i y_2 \end{aligned}\right\} \quad |z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

a) $|z-4i| + |z+4i| = 10$

$$\sqrt{x^2+(y+4)^2} + \sqrt{x^2+(y-4)^2} = 10$$

$$\left(\sqrt{x^2+(y+4)^2} + \sqrt{x^2+(y-4)^2}\right)^2 = 100$$

$$x^2+y^2+\cancel{8y}+16 + 2\sqrt{[x^2+(y+4)^2][x^2+(y-4)^2]} + x^2+y^2-\cancel{8y}+16 = 100$$

$$2x^2+2y^2+32+2\sqrt{(x^2+y^2+16+8y)(x^2+y^2+16-8y)} = 100$$

$$x^2+y^2+\sqrt{(x^2+y^2+16)^2-(8y)^2} = 34$$

$$\sqrt{(x^2+y^2+16)^2-(8y)^2} = 34-x^2-y^2$$

$$(x^2+y^2+16)^2-(8y)^2 = (34-x^2-y^2)^2 = (x^2+y^2-34)^2$$

$$\cancel{x^4}+\cancel{x^2y^2}+16x^2+\cancel{x^2y^2}+\cancel{y^4}+16y^2+16x^2+16y^2+16^2-64y^2$$

$$= \cancel{x^4}+\cancel{x^2y^2}-34x^2+\cancel{x^2y^2}+\cancel{y^4}-34y^2-34x^2-34y^2+34^2$$

$$32x^2-32y^2+256 = -68x^2-68y^2+1156$$

$$100x^2+36y^2 = 900$$

$$\frac{x^2}{9}+\frac{y^2}{25} = 1$$

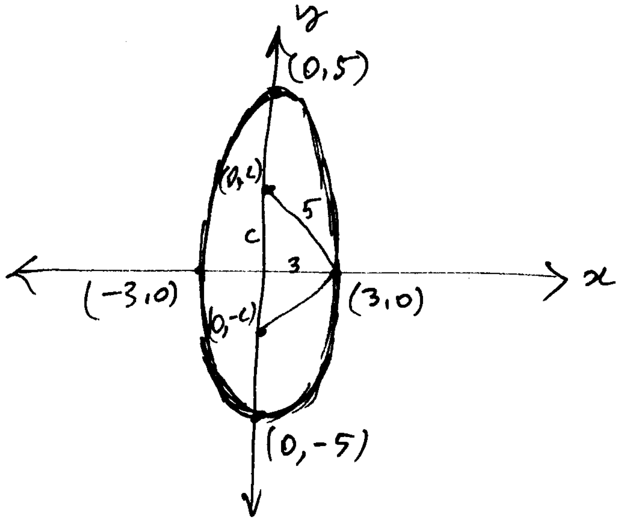
$y = 0 \Rightarrow x = \pm 3$

$x = 0 \Rightarrow y = \pm 5$

$c^2 = 25 - 9 = 16$

$c = 4$

$\therefore$ The foci are located at $(0, \pm 4)$

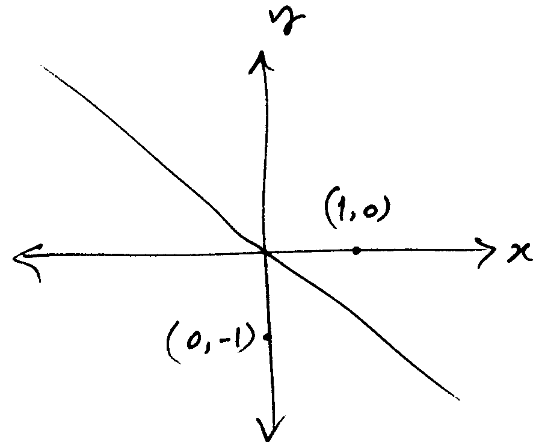


(b) $\sqrt{(x+1)^2 + y^2} = \sqrt{x^2 + (y-1)^2}$

$(x+1)^2 + y^2 = x^2 + (y-1)^2$

$\cancel{x^2} + 2x + \cancel{1} + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} - 2y + \cancel{1}$

$y = -x$



Q2

$$\begin{array}{rl} 1 + \cancel{z} + \cancel{z^2} + \cancel{\cdots} + \cancel{z^n} &= S \\ - \quad \cancel{z} + \cancel{z^2} + \cancel{z^3} + \cancel{\cdots} + z^{n+1} &= zS \\ \hline 1 - z^{n+1} &= (1-z)S \end{array} \Rightarrow S = \frac{1 - z^{n+1}}{1 - z}, (z \neq 1).$$

Substitute $z = e^{i\theta}$

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$$

$$\sum_{k=0}^{n} \cos k\theta = \sum_{k=0}^{n} Re\{e^{ik\theta}\} = Re\left\{\sum_{k=0}^{n} e^{ik\theta}\right\}$$

Therefore,

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \mathrm{Re}\left\{\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right\}$$

$$\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \frac{e^{i(n+1)\frac{\theta}{2}}\left(e^{-i(n+1)\frac{\theta}{2}}-e^{i(n+1)\frac{\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)}$$

$$= \frac{e^{i(n+1)\frac{\theta}{2}}\cdot\cancel{(-2i)}\cdot\sin(n+1)\frac{\theta}{2}}{e^{i\frac{\theta}{2}}\cdot\cancel{(-2i)}\cdot\sin\frac{\theta}{2}}$$

$$= e^{in\frac{\theta}{2}}\cdot\frac{\sin(n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\mathrm{Re}\left\{\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right\} = \mathrm{Re}\left\{e^{in\frac{\theta}{2}}\cdot\frac{\sin(n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \cos\frac{n\theta}{2}\cdot\frac{\sin(n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

{ Using the trigonometric identity $\sin a\cdot\cos b = \frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$

$a = (n+1)\frac{\theta}{2}$
$b = n\frac{\theta}{2}$ }

$$= \frac{1}{2}\,\frac{\sin\frac{\theta}{2}+\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{1}{2}+\frac{\sin(2n+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \quad //$$

Q3
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- For $n=1$

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta \quad \checkmark$$

- Assume formula is true for $n=n$.

- $$(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^n$$
$$= (\cos\theta + i\sin\theta)(\cos n\theta + i\sin n\theta)$$

$$= \cos\theta.\cos n\theta + i\cos\theta.\sin n\theta + i\sin\theta.\cos n\theta - \sin\theta.\sin n\theta$$

$$= \underbrace{\cos\theta.\cos n\theta - \sin\theta.\sin n\theta}_{\cos(n+1)\theta} + i\underbrace{(\cos\theta.\sin n\theta + \sin\theta.\cos n\theta)}_{\sin(n+1)\theta}$$

$= \cos(n+1)\theta + i\sin(n+1)\theta.$ $\checkmark$ is also true for $n+1$.

Q4
$$1 + z_0 + z_0^2 + z_0^3 + \cdots + z_0^{n-1} = \frac{1-z_0^n}{1-z_0}$$

using the identity in Q2. Since $z_0 \neq 1$,
the above equation is safe.

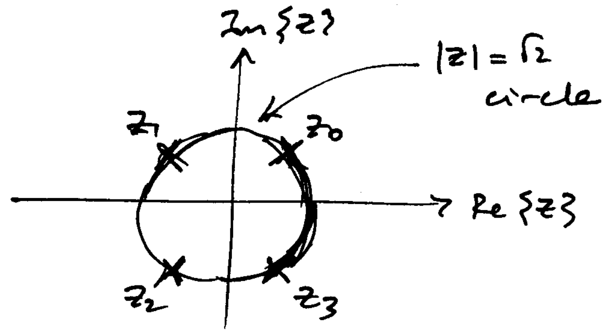
z_0 is any of the n roots of $z^n = 1$, except 1,
which means $z_0^n = 1$ also holds.

$$\therefore \frac{1-z_0^n}{1-z_0} = \frac{1-1}{1-z_0} = 0 \quad //$$

Q5 $z^4+4=0 \Rightarrow z^4=-4=4\,e^{i(\pi+2k\pi)}$ $\quad k=0,1,2,\dots$

$$z_k = \sqrt[4]{4}\ \exp\left[i\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)\right] \quad k=0,1,2,3.$$

$$z_0 = \sqrt{2}\,e^{i\pi/4} = 1+i$$
$$z_1 = \sqrt{2}\,e^{i3\pi/4} = -1+i$$
$$z_2 = \sqrt{2}\,e^{i5\pi/4} = -1-i$$
$$z_3 = \sqrt{2}\,e^{i7\pi/4} = 1-i$$



$$(z^4+4) = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$$

Since a,b,c,d are real, z_0,z_1,z_2,z_3 should be coupled in conjugate pairs. i.e., (z_0,z_3) & (z_1,z_2)

$$(z^4+4) = [(z-z_0)(z-z_3)][(z-z_1)(z-z_2)]$$
$$= [(z-1-i)(z-1+i)][(z+1-i)(z+1+i)]$$
$$= [(z-1)^2+1][(z+1)^2+1]$$
$$= (z^2-2z+1+1)(z^2+2z+1+1)$$
$$= (z^2-2z+2)(z^2+2z+2)$$

$(a=-2,\ b=2)$ & $(c=2,\ d=2)$ or vice versa.