1) $$\frac{-4\cdot(1-i)^2}{1+i} = \frac{-4\cdot(1-2\cdot i-1)}{1+i} = \frac{-4\cdot -2\cdot i}{1+i}$$

$$= \frac{8\cdot i}{1+i} = \frac{8\cdot e^{i\cdot\frac{\pi}{2}}}{\sqrt{2}\cdot e^{i\cdot\frac{\pi}{4}}} = 4\sqrt{2}\cdot e^{i\cdot\frac{\pi}{4}}$$

$$= 2^{5/2}\cdot e^{i\cdot\frac{\pi}{4}}$$

$$= 4+4\cdot i$$

$$\left(\frac{-4\cdot(1-i)^2}{1+i}\right)^{1/4} = \left(2^{5/2}\cdot e^{i\cdot\left(\frac{\pi}{4}+2\cdot k\cdot\pi\right)}\right)^{1/4}_{k=0,1,2,3}$$

$k=0 \rightarrow 2^{5/8}\cdot e^{i\cdot\frac{\pi}{16}}$ $\qquad$ $k=1 \rightarrow 2^{5/8}\cdot e^{i\cdot\frac{9\pi}{16}}$

$k=2 \rightarrow 2^{5/8}\cdot e^{i\cdot\frac{17\pi}{16}}$ $\qquad$ $k=3 \rightarrow 2^{5/8}\cdot e^{i\cdot\frac{25\pi}{16}}$

$$2)\quad -a \times 32i = a.e^{i.\pi}.2^5.e^{i.\frac{\pi}{2}} = a.2^5.e^{i.\frac{3\pi}{2}}$$

$$(-a \times 32i)^{1/5} = \underbrace{\left(a.2^5.e^{i.\left(\frac{3\pi}{2}+2.k.\pi\right)}\right)^{1/5}}_{k=0,1,2,3,4}$$

$k=0 \longrightarrow a^{1/5}.2.e^{i.\frac{3\pi}{10}}$

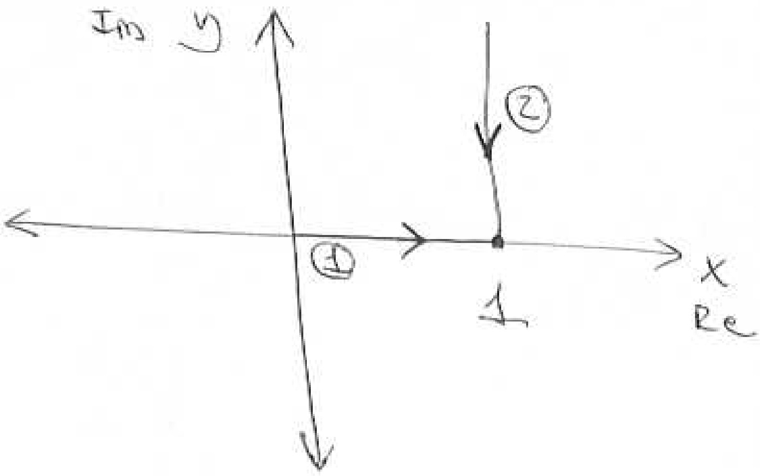
$k=1 \longrightarrow a^{1/5}.2.e^{i.\frac{7\pi}{10}}$

$k=2 \rightarrow a^{1/5}.2.e^{i.\frac{11\pi}{10}}$

$k=3 \rightarrow a^{1/5}.2.e^{i.\frac{15\pi}{10}}$

$k=4 \rightarrow a^{1/5}.2.e^{i.\frac{19\pi}{10}}$

3) $$\lim_{z \to 1} \frac{1-z}{1-\bar{z}}$$



on path 1, $z = x$

$$\lim_{z \to 1} \frac{1-z}{1-\bar{z}} = \lim_{x \to 1} \frac{1-x}{1-x} = 1$$

on path 2, $z = 1 + i \cdot y$

$$\lim_{z \to 1} \frac{1-z}{1-\bar{z}} = \lim_{y \to 0} \frac{1-(1+i \cdot y)}{1-(1-i \cdot y)} = \lim_{y \to 0} \frac{-i \cdot y}{i \cdot y}$$

$$= 1$$

The limits on path 1 and path 2 are different, so limit does not exist.

4) The definition of limit says that $\lim_{z \to z_0} f(z) = w_0$ if for each positive number ε, there is a positive number δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

If our case $f(z) = z^2$, $w_0 = z_0^2$. So;

$|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

$$|z^2 - z_0^2| \geq \left| |z|^2 - |z_0|^2 \right| < \varepsilon$$

$$\left| |z|^2 - |z_0|^2 \right| < \varepsilon \longrightarrow -\varepsilon < |z|^2 - |z_0|^2 < \varepsilon$$

$$\underbrace{|z_0|^2 - \varepsilon < |z|^2 < |z_0|^2 + \varepsilon}$$

$$|z| < \sqrt{\varepsilon + |z_0|^2}$$

$$|z - z_0| \leq |z| + |z_0| < \underbrace{|z_0| + \sqrt{\varepsilon + |z_0|^2}}_{\delta}$$

So;

$|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < |z_0| + \sqrt{\varepsilon + |z_0|^2}$

Since for every positive ε, we can find δ; the proof is finished.

5) $z = x + i \cdot y$

$$\frac{\sin z}{z} = \frac{\sin x \cdot \cosh y + i \cdot \cos x \cdot \sinh y}{x + i \cdot y}$$

$$= \frac{\sin x \cdot \cosh y \cdot x + \cos x \cdot \sinh y \cdot y + i \cdot (\cos x \cdot \sinh y \cdot x - \sin x \cdot \cosh y \cdot y)}{x^2 + y^2}$$

$$= \frac{\sin x \cdot \cosh y \cdot x + \cos x \cdot \sinh y \cdot y}{x^2 + y^2} + i \cdot \frac{\cos x \cdot \sinh y \cdot x - \sin x \cdot \cosh y \cdot y}{x^2 + y^2}$$

continuous everywhere except $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\} \Rightarrow z = 0.$

If we show that $f(z)$ is also continuous at $z = 0$, then $f(z)$ will be continuous everywhere in the complex plane.

For $f(z)$ to be continuous at $z = 0$, the following should be satisfied:

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1$$

If we can prove that $\lim_{z \to 0} \frac{\sin z}{z} = 1$, then our proof will be finished.

$$\left|\frac{\sin z}{z}-1\right|^2=\left[\frac{x\cdot\overbrace{\sin x\cdot\cosh y}^{A}+y\cdot\overbrace{\cos x\cdot\sinh y}^{B}}{x^2+y^2}-1\right]^2+\left[\frac{x\cdot\overbrace{\cos x\cdot\sinh y}^{B}-y\cdot\underbrace{\sin x\cdot\cosh y}_{A}}{x^2+y^2}\right]^2$$

$$=\left(\frac{A\cdot x+B\cdot y}{x^2+y^2}-1\right)^2+\left(\frac{B\cdot x-A\cdot y}{x^2+y^2}\right)^2$$

$$=\frac{(A\cdot x+B\cdot y)^2+(B\cdot x-A\cdot y)^2}{(x^2+y^2)^2}-2\cdot\frac{(Ax+By)}{x^2+y^2}+1$$

$$=\frac{A^2+B^2}{x^2+y^2}-2\cdot\frac{(Ax+By)}{x^2+y^2}+1$$

$$=\frac{\sin^2 x+\sinh^2 y}{x^2+y^2}-2\cdot\frac{(Ax+By)}{x^2+y^2}+1 \qquad \left(\begin{array}{l}A^2+B^2=|\sin z|^2\\ =\sin^2 x+\sinh^2 y\end{array}\right)$$

$$=\frac{x^2}{x^2+y^2}\cdot\left(\frac{\sin^2 x}{x^2}-1\right)+\frac{y^2}{x^2+y^2}\cdot\left(\frac{\sinh^2 y}{y^2}-1\right)$$

$$-2\cdot\left(\frac{A}{x}-1\right)\cdot\frac{x^2}{x^2+y^2}-2\cdot\left(\frac{B}{y}-1\right)\cdot\frac{y^2}{x^2+y^2}$$

$$\lim_{z\to 0}\left(\frac{A}{x}-1\right)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{\sin x}{x}\cdot\cosh y-1\right)=0$$

$$\lim_{z\to 0}\left(\frac{B}{y}-1\right)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{\sinh y}{y}\cdot\cos x-1\right)=0$$

$$\lim_{z\to 0}\left(\frac{\sin^2 x}{x^2}-1\right)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{\sin^2 x}{x^2}-1\right)=0$$

$$\lim_{z\to 0}\left(\frac{\sinh^2 y}{y^2}-1\right)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{\sinh^2 y}{y^2}-1\right)=0$$

So since these four limits are 0; we can say that for any $\varepsilon>0$, there exists a δ-neighborhood B of the origin so that for all $z=(x,y)\in B$ we have:

$$\left|\frac{A}{x}-1\right|<\varepsilon \qquad \left|\frac{B}{y}-1\right|<\varepsilon$$

$$\left|\frac{\sin^2 x}{x^2}-1\right|<\varepsilon \qquad \left|\frac{\sinh^2 y}{y^2}-1\right|<\varepsilon$$

$$\left|\frac{\sin z}{z}-1\right|^2<\frac{x^2}{x^2+y^2}\cdot\left|\frac{\sin^2 x}{x^2}-1\right|+\frac{y^2}{x^2+y^2}\cdot\left|\frac{\sinh^2 y}{y^2}-1\right|$$

$$+2\cdot\frac{x^2}{x^2+y^2}\cdot\left|\frac{A}{x}-1\right|+2\cdot\frac{y^2}{x^2+y^2}\cdot\left|\frac{B}{y}-1\right|$$

We know that for any $\varepsilon > 0$, there exists a δ-neighborhood B of the origin so that for all $z = (x, y) \in B$ we have:

$$\left| \frac{A}{x} - 1 \right| < \varepsilon \qquad \left| \frac{B}{y} - 1 \right| < \varepsilon$$

$$\left| \frac{\sin^2 x}{x^2} - 1 \right| < \varepsilon \qquad \left| \frac{\sinh^2 y}{y^2} - 1 \right| < \varepsilon$$

Using these:

$$\left| \frac{\sin z}{z} - 1 \right|^2 < \frac{x^2}{x^2+y^2} \cdot \varepsilon + \frac{y^2}{x^2+y^2} \cdot \varepsilon$$

$$+ 2 \cdot \frac{x^2}{x^2+y^2} \cdot \varepsilon + 2 \cdot \frac{y^2}{x^2+y^2} \cdot \varepsilon$$

$$\left| \frac{\sin z}{z} - 1 \right|^2 < 3 \cdot \varepsilon$$

$$\left| \frac{\sin z}{z} - 1 \right| < \underbrace{\sqrt{3} \cdot \sqrt{\varepsilon}}_{\varepsilon'}$$

This means that for any $\varepsilon' > 0$, there exists a δ-neighborhood B of the origin so that for all $z = (x, y) \in B$ we have:

$$\underbrace{\left| \frac{\sin z}{z} - 1 \right| < \varepsilon'}$$

$\downarrow$

This means that $\lim_{z \to 0} \frac{\sin z}{z} = 1$. So proof is finished.