E. IRCI

Math 206 — Homework #3

Solutions

1. $u_x = v_y = \sinh x \sin y$

$$v(x,y) = -\sinh x \cos y + \Phi(x)$$

$$u_y = \cosh x \cos y$$

$$v_x = -\cosh x \cos y + \Phi'(x)$$

$$u_y = -v_x \Rightarrow \Phi'(x) = 0 \rightarrow \Phi(x) = c$$

$$v(x,y) = -\sinh x \cos y + c$$

$$v(0,0) = c = 0$$

$$\therefore \boxed{v(x,y) = -\sinh x \cos y}$$

2. $u_r = \frac{1}{r} v_\theta = \frac{1}{r} \Rightarrow v_\theta = 1$

$$v(r,\theta) = \theta + \Phi(r)$$

$$u_\theta = 1 \quad , \quad v_r = \Phi'(r)$$

$$\frac{1}{r} u_\theta = -v_r \Rightarrow \frac{1}{r} = -\Phi'(r) \Rightarrow \Phi(r) = -\ln r + c$$

$$v(r,\theta) = -\ln r + \theta + c$$

$$v(1,0) = -\underbrace{\ln(1)}_{0} + 0 + c = 0 \Rightarrow c = 0$$

$$\therefore \boxed{v(r,\theta) = -\ln r + \theta}$$

3. $(1+i)^{2006} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{2006}$

$$= \sqrt{2}^{\,2006} \cdot e^{i\frac{\pi}{2}1003}$$

$$= 2^{1003} \cdot \underbrace{e^{i\frac{3\pi}{2}}}_{-i} \cdot \underbrace{e^{i\frac{\pi}{2}1000}}_{\left(e^{i2\pi}\right)^{250} = 1}$$

$$= -i\, 2^{1003}$$

$$= a + ib$$

$$\therefore \boxed{a = 0 \quad , \quad b = -2^{1003}}$$

4. a) $e^{z} = -4$

$$e^{x} e^{iy} = 4 \cdot e^{i\pi}$$

$$e^{x} = 4 \qquad y = \pi + 2n\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$x = \ln 4 \qquad y = (2n+1)\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \quad z = x + iy = \ln 4 + i(2n+1)\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

b) $e^{z} = 2 + 2i$

$$e^{x} e^{iy} = 2\sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$e^{x} = 2\sqrt{2} \qquad e^{iy} = e^{i\pi/4}$$

$$x = \ln(2\sqrt{2}) \qquad y = \frac{\pi}{4} + 2n\pi \qquad n = 0, \pm 1, \pm 2 \ldots$$

$$x = \ln(2\sqrt{2}) \qquad y = \left(2n + \frac{1}{4}\right)\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \quad z = x + iy = \ln(2\sqrt{2}) + i\left(2n + \frac{1}{4}\pi\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

c) $e^{(4z-2)} = -1$

$$e^{4x-2} e^{i4y} = 1 . e^{i\pi}$$

$$e^{4x-2} = 1 \qquad e^{i4y} = e^{i\pi}$$

$$4x-2 = \ln(1) = 0 \qquad 4y = \pi + 2n\pi \qquad n = 0, \pm 1, \pm 2, \dots$$

$$x = \frac{1}{2} \qquad y = \left(\frac{n}{2} + \frac{1}{4}\right)\pi \qquad n = 0, \pm 1, \pm 2, \dots$$

$$\Rightarrow \quad z = x + iy = \frac{1}{2} + i\left(\frac{n}{2} + \frac{1}{4}\right)\pi \qquad n = 0, \pm 1, \pm 2, \dots$$

5. a) $f(z) = u(x,y) + iv(x,y)$

- Since $f(z)$ is real-valued $v(x,y) = 0$
- Since $f(z)$ is analytic in D,

$$u_x = v_y$$
$$u_y = -v_x$$

- Since $v(x,y) = 0$

$$u_x = v_y = 0$$
$$u_y = -v_x = 0$$

which means $u(x,y)$ has no x or y dependence, i.e. $u(x,y) = c$ where c is a constant.

$$f(z) = u(x,y) = c \;,$$

which is a constant function.

b) $f(z) = u(x,y) + iv(x,y)$

$\overline{f(z)} = \tilde{u}(x,y) + i\tilde{v}(x,y) = u(x,y) - iv(x,y)$

Both $f(z)$ and $\overline{f(z)}$ is analytic in D.

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}} \qquad \boxed{\begin{aligned} \tilde{u}_x &= \tilde{v}_y \\ \tilde{u}_y &= -\tilde{v}_x \end{aligned} \Rightarrow \begin{aligned} u_x &= -v_y \\ u_y &= v_x \end{aligned}}$$

↑ for $f(z)$ ↑ for $\overline{f(z)}$

$$\left.\begin{aligned} u_x &= v_y \\ u_x &= -v_y \end{aligned}\right\} \begin{aligned} u_x &= 0 \\ v_y &= 0 \end{aligned} \qquad \left.\begin{aligned} u_y &= -v_x \\ u_y &= v_x \end{aligned}\right\} \begin{aligned} u_y &= 0 \\ v_x &= 0 \end{aligned}$$

$$\left.\begin{aligned} u_x &= 0 \\ u_y &= 0 \end{aligned}\right\} u(x,y) = c_1 \qquad \left.\begin{aligned} v_x &= 0 \\ v_y &= 0 \end{aligned}\right\} v(x,y) = c_2$$

$f(z) = u(x,y) + iv(x,y) = c_1 + ic_2$,

which is a constant function.