Math 206 - Homework #5

Solutions

1) The Cauchy integral formula states that

$$\int_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)$$

when f is analytic everywhere within and on a simple closed contour C, in the positive sense an z_0 is any point interior to C.

Therefore,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i\, e^{az_0}\Big|_{z_0=0} = 2\pi i$$

since $f(z)=e^{az}$ is analytic within and on C, C is a closed simple contour in the positive sense, $z_0=0$ is inside C.

$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{a(\cos\theta + i\sin\theta)}}{\cancel{e^{i\theta}}} \cdot \cancel{i}\cancel{e^{i\theta}}\,d\theta = 2\pi \cancel{i}$$

since $z = e^{i\theta} = \cos\theta + i\sin\theta$

$dz = i e^{i\theta}\,d\theta$

C is the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$.

Therefore,

$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cdot e^{ia\sin\theta}\,d\theta = 2\pi$$

$$\int_{-\pi}^{\pi} \left[e^{a\cos\theta} \cdot \cos(a\sin\theta) + i\, e^{a\cos\theta} \sin(a\sin\theta)\right] d\theta = 2\pi$$

Taking the real parts of both sides,

$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cdot \cos(a\sin\theta)\, d\theta = 2\pi$$

Since $e^{a\cos\theta} \cdot \cos(a\sin\theta)$ is an even function of θ,

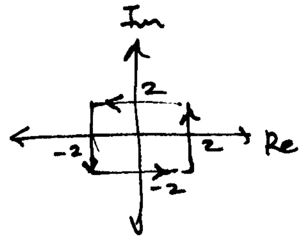
$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cdot \cos(a\sin\theta)\, d\theta = 2\int_{0}^{\pi} e^{a\cos\theta} \cdot \cos(a\sin\theta)\, d\theta = 2\pi$$

which gives

$$\int_{0}^{\pi} e^{a\cos\theta} \cdot \cos(a\sin\theta)\, d\theta = \pi$$

∎

2) The contour C can be sketched as:



In this question we will use the extended Cauchy integral formula

$$\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \dots)$$

which reduces to the simple Cauchy integral formula in Question 1 when $n=0$.

a) $\int_C \frac{\cos z}{z(z^2+8)} dz = \int_C \frac{\frac{\cos z}{z^2+8}}{(z-0)^1} dz$, $f(z) = \frac{\cos z}{z^2+8}$ ← analytic in and on C. ✓

$z_0 = 0$, (inside C) ✓

$f(z_0) = \frac{1}{8}$

$\therefore \int_C \frac{\cos z}{z(z^2+8)} dz = \frac{2\pi i}{0!} f^{(0)}(z_0)$

$= 2\pi i \cdot \frac{1}{8} = \underline{\underline{\frac{\pi}{4} i}}$

b) $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = ?$ $(-2 < x_0 < 2)$

$f(z) = \tan(z/2)$ ← analytic in and on C. ✓

$z_0 = x_0$, (inside C) ✓

$\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = \frac{2\pi i}{1!} f^{(1)}(x_0)$

$f^{(1)}(z) \Big|_{z=x_0} = \frac{1}{2} \sec^2(z/2) \Big|_{z=x_0} = \frac{1}{2} \sec^2(x_0/2)$

$\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz = 2\pi i \cdot \frac{1}{2} \sec^2(x_0/2) = \underline{\underline{\pi i \sec^2\left(\frac{x_0}{2}\right)}}$

c) $\int_C \frac{\cosh z}{z^4} dz = ?$

$f(z) = \cosh z$ ← analytic inside and on C. ✓

$z_0 = 0$, (inside C) ✓

$= \int_C \frac{\cosh z}{(z-0)^4} = \frac{2\pi i}{3!} f^{(3)}(0)$

$f(z) = \cosh z$

$f^{(1)}(z) = \sinh z$

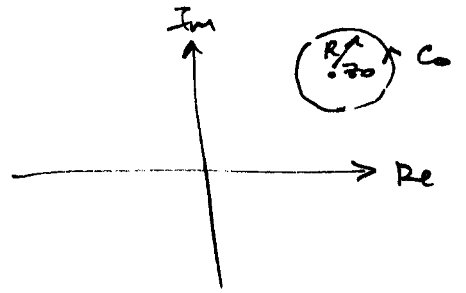
$f^{(2)}(z) = \cosh z$

$f^{(3)}(z) = \sinh z$.

$= \frac{2\pi i}{6} \cdot \sinh(0) = \underline{\underline{0}}$

$\sinh(0) = 0$.

3)



$$\int_{C_0} (z-z_0)^{n-1}\,dz = ?$$

- When $n \geq 1$, $f(z) = (z-z_0)^{n-1}$ is analytic at all points interior and on C_0. Therefore, utilizing the Cauchy-Goursat Theorem, the integral is 0.

- When $n=0$

$$\int_{C_0} (z-z_0)^{-1}\,dz = \int_{C_0} \frac{1}{z-z_0}\,dz = 2\pi i\, f(z_0) \qquad \text{where } f(z)=1,\ f(z_0)=1$$

$$= \underline{\underline{2\pi i}}$$

- When $n \leq -1$

$$\int_{C_0} (z-z_0)^{n-1}\,dz = \int_{C_0} \frac{1}{(z-z_0)^{1-n}}\,dz = \frac{2\pi i}{(-n)!} f^{(-n)}(z_0) \qquad \text{where } f(z)=1$$

$$f^{(1)}(z) = 0$$

$$f^{(2)}(z) = 0$$

$$\vdots$$

$$f^{(-n)}(z) = 0$$

$$= \underline{\underline{0}}$$

■

4) $$g(w) = \int_C \frac{2z^2-z-2}{z-w}\,dz = 2\pi i\, f(w) \quad , \ \overset{\text{analytic inside and on } C.}{f(z) = 2z^2-z-2}$$

$w = 2$, inside C.

$$g(2) = 2\pi i \cdot (2z^2-z-2)\Big|_{z=2} = 8\pi i$$

When $|w|>3$ the function $\frac{2z^2-z-2}{z-w}$ is analytic interior and on $C = |z|=3$, so by the Cauchy-Goursat Theorem, $g(w)=0$ when $|w|>3$. ■

5)

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| \le M_R \cdot L$$

L: length of contour $C_R = 2\pi R$

M_R: upper-bound for $|f(z)| = \left|\frac{\operatorname{Log} z}{z^2}\right|$ on C_R.

$\therefore$ for all z on C_R

$$\left|\frac{\operatorname{Log} z}{z^2}\right| \le M_R \quad , \quad \left|\frac{\operatorname{Log} z}{z}\right| = \frac{|\operatorname{Log} z|}{|z|^2}$$

C_R: $z = Re^{i\theta}$, $-\pi \le \theta \le \pi$, $R > 1$.

on C_R $\operatorname{Log} z = \ln R + i\theta$

$$\left|\frac{\operatorname{Log} z}{z^2}\right| = \frac{|\operatorname{Log} z|}{|z|^2} \le \underbrace{\frac{\sqrt{(\ln R)^2 + \pi^2}}{R^2}}_{M_R}$$

since $\max(|\theta|) = \pi$, $\max(\theta^2) = \pi^2$

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| \le 2\pi \cancel{R} \cdot \frac{\sqrt{(\ln R)^2 + \pi^2}}{R^{\cancel{2}}} = 2\pi \frac{\sqrt{(\ln R)^2 + \pi^2}}{R}$$

$$\sqrt{(\ln R)^2 + \pi^2} < \sqrt{(\ln R)^2 + 2\pi \ln R + \pi^2} = \ln R + \pi$$

(since $R > 1$, $\ln R > 0$, $\pi > 0$)

$$\therefore 0 \le \left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \frac{\ln R + \pi}{R}$$

$\hookrightarrow$ goes to 0 as $R \to \infty$.

$$\lim_{R \to \infty} 2\pi \frac{\ln R + \pi}{R} = \lim_{R \to \infty} 2\pi \frac{1/R}{1} = 0$$

$$\therefore \lim_{R \to \infty} \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz = 0$$

■