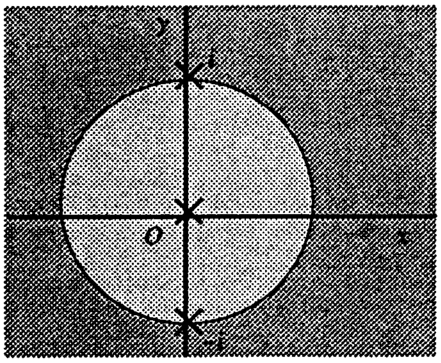
Math 206 - Homework #7
Solutions

Q1. The function $f(z)=\dfrac{1}{z(1+z^2)}$ has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below. Hence there is a Laurent series representation for the domain $0<|z|<1$ and also one for the domain $1<|z|<\infty$, which is exterior to the circle $|z|=1$.



To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1).$$

For the domain $0<|z|<1$, we have

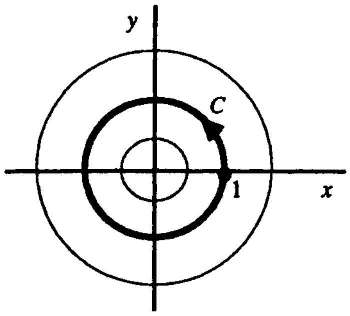
$$f(z)=\frac{1}{z}\cdot\frac{1}{1+z^2}=\frac{1}{z}\sum_{n=0}^{\infty}\left(-z^2\right)^n=\sum_{n=0}^{\infty}(-1)^n z^{2n-1}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}=\sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1}+\frac{1}{z}.$$

On the other hand, when $1<|z|<\infty$,

$$f(z)=\frac{1}{z^3}\cdot\frac{1}{1+\frac{1}{z^2}}=\frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

In this second expansion, we have used the fact that $(-1)^{n-1}=(-1)^{n-1}(-1)^2=(-1)^{n+1}$.

Q2. *(a)* The function $f(z)$ is analytic in some annular domain centered at the origin; and the unit circle $C: z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$ is contained in that domain, as shown below.



For each point z in the annular domain, there is a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n},$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^{n+1}} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(n+1)}} ie^{i\phi} d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi}) e^{-in\phi} d\phi \qquad (n = 0,1,2,\ldots)$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^{-n+1}} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(-n+1)}} ie^{i\phi} d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi}) e^{in\phi} d\phi \qquad (n = 1,2,\ldots).$$

Substituting these values of a_n and b_n into the series, we then have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi}) e^{-in\phi} d\phi \, z^n + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi}) e^{in\phi} d\phi \, \frac{1}{z^n},$$

or

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] d\phi.$$

(b) Put $z = e^{i\theta}$ in the final result in part *(a)* to get

$$f(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right]d\phi,$$

or

$$f(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\cos[n(\theta-\phi)]d\phi.$$

If $u(\theta) = \operatorname{Re} f(e^{i\theta})$, then, equating the real parts on each side of this last equation yields

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]d\phi.$$

Q3. Let C denote the circle $|z| = 1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ $(|z| < \infty)$ enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right)dz = \int_C e^z e^{1/z}dz = \int_C e^{1/z}\sum_{n=0}^{\infty}\frac{z^n}{n!}\,dz = \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n \exp\left(\frac{1}{z}\right)dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{z^k} = \sum_{k=0}^{\infty}\frac{1}{k!}z^{n-k} \qquad (n = 0,1,2,\ldots).$$

Now the $\frac{1}{z}$ in this series occurs when $n - k = -1$, or $k = n+1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right)dz = 2\pi i\frac{1}{(n+1)!} \qquad (n = 0,1,2,\ldots).$$

The final result in part *(a)* thus reduces to

$$\int_C \exp\left(z + \frac{1}{z}\right)dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}.$$

Q4. We need to find the first four nonzero coefficients in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \qquad \left(|z| < \frac{\pi}{2}\right).$$

This representation is valid in the stated disk since the zeros of $\cosh z$ are the numbers $z = \left(\frac{\pi}{2} + n\pi\right)i$ $(n = 0, \pm 1, \pm 2, \ldots)$, the ones nearest to the origin being $z = \pm\frac{\pi}{2}i$. The series contains only even powers of z since $\cosh z$ is an even function; that is, $E_{2n+1} = 0$ $(n = 0, 1, 2, \ldots)$. To find the series, we divide the series

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots = 1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \cdots \qquad (|z| < \infty)$$

into 1. The result is

$$\frac{1}{\cosh z} = 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right)$$

or

$$\frac{1}{\cosh z} = 1 - \frac{1}{2!}z^2 + \frac{5}{4!}z^4 - \frac{61}{6!}z^6 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right).$$

Since

$$\frac{1}{\cosh z} = E_0 + \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 + \frac{E_6}{6!}z^6 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right),$$

this tells us that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$

Q5. Since $f(z)$ is analytic at z_0; we can represent $f(z)$ by a Taylor series for $0 \leq |z-z_0| < R$ and the Taylor series will converge to $f(z)$ for all points inside the circle $|z-z_0| = R$. Here R is the minimum distance between z_0 and a point z_1 where $f(z)$ fails to be analytic. So;

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0)^n = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0)^n;$$

since $f(z_0) = f'(z_0) = \dots = f^{(m)}(z_0) = 0$ $\quad (0 \leq |z-z_0| < R)$

Using this; $g(z)$ has the series representation

$$g(z) = \frac{f(z)}{(z-z_0)^{m+1}} = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0)^{n-(m+1)}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n+m+1)}(z_0)}{(n+m+1)!} \cdot (z-z_0)^n \quad \text{in the disk}$$

$$0 < |z-z_0| < R.$$

At $z = z_0$, the series gives:

$$\sum_{n=0}^{\infty} \frac{f^{(n+m+1)}(z_0)}{(n+m+1)!} \cdot (z-z_0)^n \Bigg|_{z=z_0} = \left(\frac{f^{(m+1)}(z_0)}{(m+1)!} + \frac{f^{(m+2)}(z_0)}{(m+2)!} \cdot (z-z_0) + \frac{f^{(m+3)}(z_0)}{(m+3)!} \cdot (z-z_0)^2 + \dots \right) \Bigg|_{z=z_0} = \frac{f^{(m+1)}(z_0)}{(m+1)!}$$

This is $g(z_0)$.

We know that the series converges to $g(z)$ for $0 < |z - z_0| < R$.

But we notice that it also converges to $g(z)$ for $z = z_0$. So $g(z)$ has the series representation

$$\sum_{n=0}^{\infty} \frac{f^{(n+m+1)}(z_0)}{(n+m+1)!} \cdot (z - z_0)^n \quad \text{for } 0 \leq |z - z_0| < R$$

Since $g(z)$ has a convergent series representation at z_0 and in a circle centered at z_0, it is analytic at $z = z_0$.